

# Nonparametric Regression

Appendix to *An R and S-PLUS Companion to Applied Regression*

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## 1 Nonparametric Regression Models

The traditional nonlinear regression model (described in the Appendix on nonlinear regression) fits the model

$$y_i = f(\boldsymbol{\beta}, \mathbf{x}'_i) + \varepsilon_i$$

where  $\boldsymbol{\beta} = (\beta_1, \dots, \beta_p)'$  is a vector of parameters to be estimated, and  $\mathbf{x}'_i = (x_1, \dots, x_k)$  is a vector of predictors for the  $i$ th of  $n$  observations; the errors  $\varepsilon_i$  are assumed to be normally and independently distributed with mean 0 and constant variance  $\sigma^2$ . The function  $f(\cdot)$ , relating the average value of the response  $y$  to the predictors, is specified in advance, as it is in a linear regression model.

The general nonparametric regression model is written in a similar manner, but the function  $f$  is left unspecified:

$$\begin{aligned} y_i &= f(\mathbf{x}'_i) + \varepsilon_i \\ &= f(x_{i1}, x_{i2}, \dots, x_{ik}) + \varepsilon_i \end{aligned}$$

Moreover, the object of nonparametric regression is to estimate the regression function  $f(\cdot)$  directly, rather than to estimate parameters. Most methods of nonparametric regression implicitly assume that  $f(\cdot)$  is a smooth, continuous function.<sup>1</sup> As in nonlinear regression, it is standard to assume that  $\varepsilon_i \sim \text{NID}(0, \sigma^2)$ .

An important special case of the general model is nonparametric simple regression, where there is only one predictor:

$$y_i = f(x_i) + \varepsilon_i$$

Nonparametric simple regression is often called ‘scatterplot smoothing’ because an important application is to tracing a smooth curve through a scatterplot of  $y$  against  $x$ . I frequently used nonparametric regression in this manner in the body of the text.

Because it is difficult to fit the general nonparametric regression model when there are many predictors, and because it is difficult to display the fitted model when there are more than two or three predictors, more restrictive models have been developed. One such model is the *additive regression model*,

$$y_i = \alpha + f_1(x_{i1}) + f_2(x_{i2}) + \dots + f_k(x_{ik}) + \varepsilon_i$$

where the partial-regression functions  $f_j(\cdot)$  are assumed to be smooth, and are to be estimated from the data. This model is substantially more restrictive than the general nonparametric regression model, but less restrictive than the linear regression model, which assumes that all of the partial-regression functions are linear.

Variations on the additive regression model include semiparametric models, in which some of the predictors enter linearly, for example,

$$y_i = \alpha + \beta_1 x_{i1} + f_2(x_{i2}) + \dots + f_k(x_{ik}) + \varepsilon_i$$

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<sup>1</sup>An exception to the implicit assumption of smoothness is wavelet regression, not discussed in this appendix, which is implemented in S in the `wavethresh` library; see Nason and Silveanu (1994, 2000).

(particularly useful when some of the predictors are factors), and models in which some predictors enter into interactions, which appear as higher-dimensional terms in the model, for example,

$$y_i = \alpha + f_{12}(x_{i1}, x_{i2}) + f_3(x_{i3}) + \cdots + f_k(x_{ik}) + \varepsilon_i$$

All of these models extend straightforwardly to *generalized nonparametric regression*, much as linear models extend to generalized linear models (discussed in Chapter 5 of the text). The random and link components are as in generalized linear models, but the linear predictor of the GLM

$$\eta_i = \alpha + \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_k x_{ik}$$

is replaced, for example, by an unspecified smooth function of the predictors

$$\eta_i = f(x_{i1}, x_{i2}, \dots, x_{ik})$$

for the most general case, or by a sum of smooth partial-regression functions

$$\eta_i = \alpha + f_1(x_{i1}) + f_2(x_{i2}) + \cdots + f_k(x_{ik})$$

for the *generalized additive model*.

All of these nonparametric regression models (and some others, such as *projection-pursuit regression*, and *classification and regression trees*) are discussed in Fox (2000a, 2000b), from which the examples appearing below are adapted.

## 2 Estimation

There are several approaches to estimating nonparametric regression models, of which I will describe two: local polynomial regression and smoothing splines. With respect to implementation of these methods in S, there is an embarrassment of riches:

- Local polynomial regression is performed by the standard S functions `lowess` (locally weighted scatterplot smoother, for the simple-regression case) and `loess` (local regression, more generally); the latter is in the R `modreg` library, which is part of the standard R distribution.
- Simple-regression smoothing-spline estimation is performed by the standard S function `smooth.spline` (also in the R `modreg` library).
- Generalized nonparametric regression by local likelihood estimation (of which local regression is a special case for models with normal errors) is implemented in the `locfit` (local fitting) library (Loader, 1999), which also performs density estimation.
- Generalized additive models in S-PLUS may be fit with Hastie and Tibshirani's (1990) `gam` function, which uses spline or local-regression smoothers; in R, the `gam` function from Wood's (2000, 2001) `mgcv` library fits this class of models, using spline smoothers, and featuring automatic selection of smoothing parameters. (The name of the library comes from the method employed to pick the smoothing parameters: multiple generalized cross-validation.)
- There are several other S libraries for nonparametric regression, including Bowman and Azzalini's (1997) `sm` (smoothing) library, available for both R and S-PLUS, which performs local-regression and local-likelihood estimation, and which also includes facilities for nonparametric density estimation; and Gu's (2000) `gss` (general smoothing splines) library for R, which fits various smoothing-spline regression and generalized regression models.

## 2.1 Local Polynomial Regression

### 2.1.1 Simple Regression

Here, we are looking to fit the model

$$y_i = f(x_i) + \varepsilon_i$$

Let us focus on evaluating the regression function at a particular  $x$ -value,  $x_0$ . (Ultimately, we will fit the model at a representative range of values of  $x$  or simply at the  $n$  observations,  $x_i$ .) We proceed to perform a  $p$ th-order weighted-least-squares polynomial regression of  $y$  on  $x$ ,

$$y_i = a + b_1(x_i - x_0) + b_2(x_i - x_0)^2 + \cdots + b_p(x_i - x_0)^p + e_i$$

weighting the observations in relation to their proximity to the focal value  $x_0$ ; a common weight function to use is the *tricube function*:

$$W(z) = \begin{cases} (1 - |z|^3)^3 & \text{for } |z| < 1 \\ 0 & \text{for } |z| \geq 1 \end{cases}$$

In the present context,  $z_i = (x_i - x_0)/h$ , where  $h$  is the half-width of a window enclosing the observations for the local regression. The fitted value at  $x_0$ , that is, the estimated height of the regression curve, is simply  $\hat{y}_0 = a$  (produced conveniently by having centered the predictor  $x$  at the focal value  $x_0$ ).

It is typical to adjust  $h$  so that each local regression includes a fixed proportion  $s$  of the data; then,  $s$  is called the span of the local-regression smoother. The larger the span, the smoother the result; conversely, the larger the order of the local regressions  $p$ , the more flexible the smooth.

The process of fitting a local regression is illustrated in Figure 1, using the Canadian occupational-prestige data introduced in Chapter 2 of the text. I examine the regression of **prestige** on **income**, focusing initially on the observation with the 80th largest **income** value,  $x_{(80)}$ , represented in Figure 1 by the vertical solid line.

- A window including the 50 nearest  $x$ -neighbors of  $x_{(80)}$  (i.e., for span  $s = 50/102 \simeq 1/2$ ) is shown in Figure 1 (a).
- The tricube weights for observations in this neighborhood appear in Figure 1 (b).
- Figure 1 (c) shows the locally weighted regression line fit to the data in the neighborhood of  $x_0$  (i.e., a local polynomial regression of order  $p = 1$ ); the fitted value  $\hat{y}|_{x_{(80)}}$  is represented in this graph as a larger solid dot.
- Finally, in Figure 1 (d), local regressions are estimated for a range of  $x$ -values, and the fitted values are connected in a nonparametric-regression curve.

Figure 1 (d) is produced by the following S commands, using the `lowess` function:

```
> library(car) # for data sets
. . .
> data(Prestige)
> attach(Prestige)
> plot(income, prestige, xlab="Average Income", ylab="Prestige")
> lines(lowess(income, prestige, f=0.5, iter=0), lwd=2)
>
```

The argument `f` to `lowess` gives the span of the local-regression smoother; `iter=0` specifies that the local regressions should *not* be refit to downweight outlying observations.<sup>2</sup>

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<sup>2</sup>By default, `lowess` performs `iter=3` ‘robustness’ iterations, using a bisquare weight function. The idea of weighting observations to obtain robust regression estimators is described in the Appendix on robust regression.

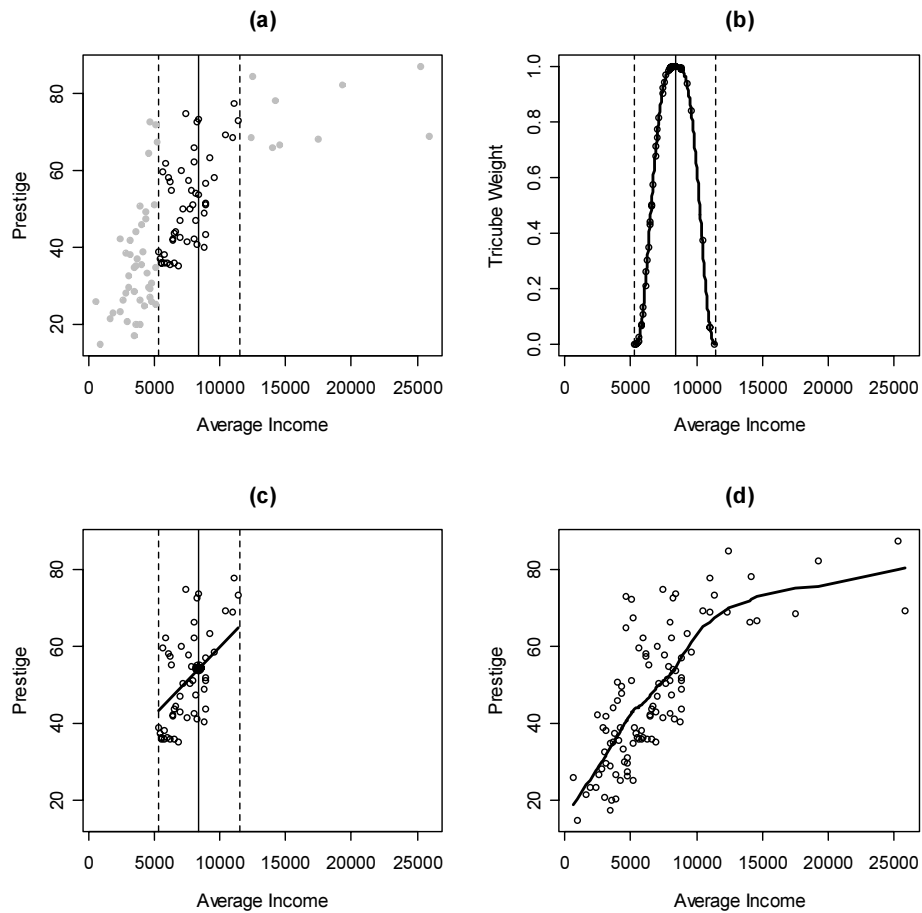


Figure 1: Local linear regression of **prestige** on **income** for the Canadian occupational-prestige data: (a) The broken lines delimit the 50 nearest neighbors of  $x_{(80)}$  (at the solid vertical line). (b) Tricube weights for observations in the neighborhood of  $x_{(80)}$ . (c) Locally weighted linear regression in the neighborhood of  $x_{(80)}$ ; the solid dot is the fitted value above  $x_{(80)}$ . (d) The completed locally linear regression, connecting fitted values across the range of  $x$ .

### 2.1.2 Multiple Regression

The nonparametric multiple regression model is

$$\begin{aligned}y_i &= f(\mathbf{x}'_i) + \varepsilon_i \\ &= f(x_{i1}, x_{i2}, \dots, x_{ik}) + \varepsilon_i\end{aligned}$$

Extending the local-polynomial approach to multiple regression is simple conceptually, but can run into practical difficulties.

- The first step is to define a multivariate neighborhood around a focal point  $\mathbf{x}'_0 = (x_{01}, x_{02}, \dots, x_{0k})$ . The default approach in the `loess` function is to employ scaled Euclidean distances:

$$D(\mathbf{x}_i, \mathbf{x}_0) = \sqrt{\sum_{j=1}^k (z_{ij} - z_{0j})^2}$$

where the  $z_j$  are the standardized predictors,

$$z_{ij} = \frac{x_{ij} - \bar{x}_j}{s_j}$$

Here  $\bar{x}_j$  is the mean of the  $j$ th predictor and  $s_j$  is its standard deviation.

- Weights are defined using the scaled distances:

$$w_i = W \left[ \frac{D(\mathbf{x}_i, \mathbf{x}_0)}{h} \right]$$

where  $W(\cdot)$  is a suitable weight function, such as the tricube, in which case  $h$  is the half-width (i.e., radius) of the neighborhood. As in local simple regression,  $h$  may be adjusted to define a neighborhood including the  $[ns]$  nearest neighbors of  $\mathbf{x}_0$  (where the square brackets denote rounding to the nearest integer).

- Perform a weighted polynomial regression of  $y$  on the  $x$ 's; for example, a local linear fit takes the following form:

$$y_i = a + b_1(x_{i1} - x_{01}) + b_2(x_{i2} - x_{02}) + \dots + b_k(x_{ik} - x_{0k}) + e_i$$

The fitted value at  $\mathbf{x}_0$  is then simply  $\hat{y}_0 = a$ .

- The procedure is repeated for representative combinations of predictor values to build up a picture of the regression surface.

Extending the illustration of the previous section, and using the `loess` function, let us regress `prestige` on both the `income` and `education` levels of the occupations:

```
> library(modreg)
> mod.lo <- loess(prestige ~ income + education, span=.5, degree=1)
> summary(mod.lo)
Call:
loess(formula = prestige ~ income + education, span = 0.5, degree = 1)
```

```
Number of Observations: 102
Equivalent Number of Parameters: 8.03
Residual Standard Error: 6.91
Trace of smoother matrix: 10.5
```

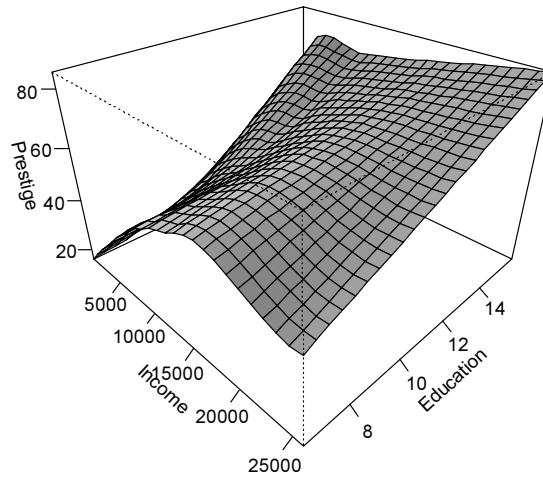


Figure 2: Fitted surface for the local linear multiple regression of `prestige` on `income` and `education`.

```
Control settings:
normalize: TRUE
span      : 0.5
degree   : 1
family    : gaussian
surface   : interpolate      cell = 0.2
```

Specifying `degree=1` fits locally linear regressions; the default is `degree=2` (i.e., locally quadratic regressions). To see the full range of arguments for the `loess` function, consult the on-line help. The `summary` output includes the standard deviation of the residuals under the model and an estimate of the equivalent number of parameters (or degrees of freedom) employed by the model — in this case, about 8 parameters. In contrast, a standard linear regression model would have employed 3 parameters (the constant and two slopes).

As in nonparametric simple regression, there are no parameters estimates: To see the result of the regression, we have to examine the fitted regression surface graphically, as in Figure 2, produced by the following S statements:<sup>3</sup>

```
> inc <- seq(min(income), max(income), len=25)
> ed <- seq(min(education), max(education), len=25)
> newdata <- expand.grid(income=inc, education=ed)
> fit.prestige <- matrix(predict(mod.lo, newdata), 25, 25)
> persp(inc, ed, fit.prestige, theta=45, phi=30, ticktype='detailed',
+       xlab='Income', ylab='Education', zlab='Prestige', expand=2/3,
+       shade=0.5)
>
```

---

<sup>3</sup>There are alternative graphical representations of the regression surface, such as contour plots and coplots. The latter can be employed when there are more than two predictors.

I employ the `expand.grid` function to create a data frame containing combinations of values of the two predictors, `income` and `education`; for each predictor, I take 25 values, evenly spaced along the range of the variable. Then, corresponding fitted values on the regression surface are computed by `predict`. These predicted values are reshaped into a 25 by 25 matrix, which is passed to the `persp` function, along with the values of the predictors (`inc` and `ed`) used to generate the regression surface.

The relationship of `prestige` to `education` and `income` appears to be nonlinear, especially in the direction of `income` (look at the grid lines on the regression surface). The partial regression in the direction of each predictor does not appear to change very much as the other predictor varies, suggesting that an additive model may be appropriate for these data. I consider such a model below.

We can also address the statistical significance of each predictor by dropping it from the model and performing an approximate incremental  $F$ -test for the change in the residual sum of squares. In fitting these separate models, I set the span of the local simple regressions to  $0.7 \simeq \sqrt{0.5}$ :

```
> mod.lo.inc <- loess(prestige ~ income, span=.7, degree=1) # omitting education
> mod.lo.ed <- loess(prestige ~ education, span=.7, degree=1) # omitting income

> anova(mod.lo.inc, mod.lo) # test for education
Model 1: loess(formula = prestige ~ income, span = 0.7, degree = 1)
Model 2: loess(formula = prestige ~ income + education, span = 0.5, degree = 1)
```

Analysis of Variance: denominator df 90.66

ENP	RSS	F-value	Pr(>F)
1	4	12006	
2	8	4246	21 4.4e-16

```
> anova(mod.lo.ed, mod.lo) # test for income
Model 1: loess(formula = prestige ~ education, span = 0.7, degree = 1)
Model 2: loess(formula = prestige ~ income + education, span = 0.5, degree = 1)
```

Analysis of Variance: denominator df 90.66

ENP	RSS	F-value	Pr(>F)
1	3	7640	
2	8	4246	8 7.1e-08

Both `income` and `education`, therefore, have highly statistically significant effects.

## 2.2 Smoothing Splines

*Smoothing splines* arise as the solution to the following simple-regression problem: Find the function  $\hat{f}(x)$  with two continuous derivatives that minimizes the *penalized sum of squares*,

$$SS^*(h) = \sum_{i=1}^n [y_i - f(x_i)]^2 + h \int_{x_{\min}}^{x_{\max}} [f''(x)]^2 dx \quad (1)$$

where  $h$  is a smoothing parameter, analogous to the neighborhood-width of the local-polynomial estimator.

- The first term in Equation (1) is the residual sum of squares.
- The second term is a *roughness penalty*, which is large when the integrated second derivative of the regression function  $f''(x)$  is large — that is, when  $f(x)$  is ‘rough’ (with rapidly changing slope). The endpoints of the integral enclose the data.

- At one extreme, when the smoothing constant is set to  $h = 0$  (and if all the  $x$ -values are distinct),  $\hat{f}(x)$  simply interpolates the data; this is similar to a local-regression estimate with  $\text{span} = 1/n$ .
- At the other extreme, if  $h$  is very large, then  $\hat{f}$  will be selected so that  $\hat{f}''(x)$  is everywhere 0, which implies a globally linear least-squares fit to the data (equivalent to local regression with infinite neighborhoods).

The function  $\hat{f}(x)$  that minimizes Equation (1) is a natural cubic spline with knots at the distinct observed values of  $x$ .<sup>4</sup> Although this result seems to imply that  $n$  parameters are required (when all  $x$ -values are distinct), the roughness penalty imposes additional constraints on the solution, typically reducing the *equivalent number of parameters* for the smoothing spline substantially, and preventing  $\hat{f}(x)$  from interpolating the data. Indeed, it is common to select the smoothing parameter  $h$  indirectly by setting the equivalent number of parameters for the smoother.

Because there is an explicit objective-function to optimize, smoothing splines are more elegant mathematically than local regression. It is more difficult, however, to generalize smoothing splines to multiple regression,<sup>5</sup> and smoothing-spline and local-regression fits with the same equivalent number of parameters are usually very similar.

An illustration appears in Figure 3, comparing a smoothing spline with a local-linear fit employing the same number of equivalent parameters (degrees of freedom). I use the `smooth.spline` function (in the `modreg` library in R) along with a previous `loess` model to show alternative fits (each with 3.85 equivalent parameters) to the relationship of `prestige` to `income`:

```
> mod.lo.inc # previously fit loess model
Call:
loess(formula = prestige ~ income, span = 0.7, degree = 1)

Number of Observations: 102
Equivalent Number of Parameters: 3.85
Residual Standard Error: 11.1

> plot(income, prestige)
> inc.100 <- seq(min(income), max(income), len=100) # 100 x-values
> pres <- predict(mod.lo.inc, data.frame(income=inc.100)) # fitted values
> lines(inc.100, pres, lty=2, lwd=2) # loess curve
> lines(smooth.spline(income, prestige, df=3.85), lwd=2) # smoothing spline
>
```

Note how I graph the local-linear regression by using `predict` to calculate 100 fitted values over the range of `income`. The two smooths are very similar: The solid line is the local-linear fit; the broken line is the smoothing spline.

## 2.3 Selecting the Smoothing Parameter

Both local-polynomial regression and smoothing splines have an adjustable smoothing parameter. This parameter may be selected by visual trial and error, picking a value that balances smoothness against fidelity to the data. More formal methods of selecting smoothing parameters typically try to minimize the mean-squared error of the fit, either by employing a formula approximating the mean-square error (e.g., so-called ‘plug-in’ estimates), or by some form of cross-validation.

In cross-validation, the data are divided into subsets (possibly comprising the individual observations); the model is successively fit omitting each subset in turn; and then the fitted model is used to ‘predict’ the response for the left-out subset. Trying this procedure for different values of the smoothing parameter

<sup>4</sup>*Splines* are piece-wise polynomial functions that fit together (at ‘knots’); for cubic splines, the first and second derivatives are also continuous at the knots. Natural splines place two additional knots at the ends of the data, and constrain the function to be linear beyond these points.

<sup>5</sup>More complicated variants, such as *thin-plate splines*, generalize more easily to multiple regression. See, e.g., Gu (2000).

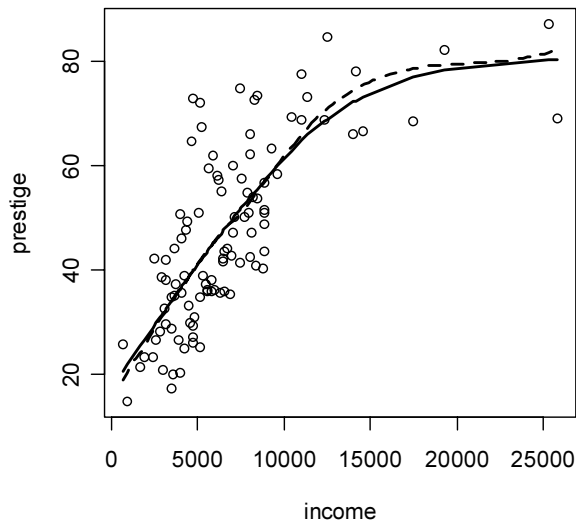


Figure 3: Local-regression (solid line) and smoothing-spline (broken line) fits for the regression of `prestige` on `income`. Both models employ 3.85 equivalent parameters.

will suggest a value that minimizes the cross-validation estimate of the mean-squared error. Because cross-validation is very computationally intensive, approximations are often employed.

## 2.4 Additive Nonparametric Regression

The additive nonparametric regression model is

$$y_i = \alpha + f_1(x_{i1}) + f_2(x_{i2}) + \dots + f_k(x_{ik}) + \varepsilon_i$$

where the partial-regression functions  $f_j(\cdot)$  are fit using a simple-regression smoother, such as local polynomial regression or smoothing splines. I illustrate for the regression of `prestige` on `income` and `education`, employing the `gam` function in the `mgcv` library (Wood, 2000, 2001):

```
> library(mgcv)
This is mgcv 0.6.2
> mod.gam <- gam(prestige ~ s(income) + s(education))
> mod.gam
```

```
Family: gaussian
Link function: identity
```

```
Formula:
prestige ~ s(income) + s(education)
```

```
Estimated degrees of freedom:
3.1178 3.1773 total = 7.2951
```

```
GCV score: 52.143
```

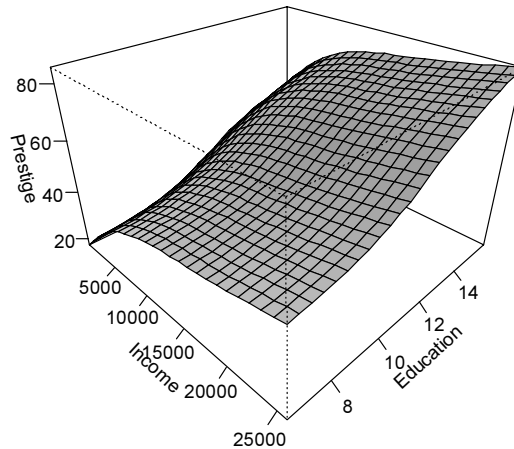


Figure 4: Fitted surface for the additive nonparametric regression of `prestige` on `income` and `education`.

The `s` function, used in specifying the model formula, indicates that each term is to be fit with a smoothing spline. The degrees of freedom for each term are found by generalized cross validation:<sup>6</sup> In this case, the equivalent of 3.1178 parameters are used for the `income` term, and 3.1773 for the `education` term; the degrees of freedom for the model are the sum of these plus 1, for the regression constant.

The additive regression surface is plotted in Figure 4:

```
> fit.prestige <- matrix(predict(mod.gam, newdata), 25, 25)
> persp(inc, ed, fit.prestige, theta=45, phi=30, ticktype='detailed',
+       xlab='Income', ylab='Education', zlab='Prestige', expand=2/3,
+       shade=0.5)
>
```

The data frame `newdata`, used to find predicted values on the regression surface, was calculated earlier to draw Figure 2 for the *general* nonparametric multiple-regression model fit to these data. The two fits are quite similar. Moreover, because slices of the additive-regression surface in the direction of one predictor (holding the other predictor constant) are parallel, it suffices to graph each partial-regression function separately. This is the practical virtue of the additive-regression model: It reduces a multidimensional (in this case, only three-dimensional) regression problem to a series of two-dimensional partial-regression graphs. The `plot` method for `gam` objects produces these graphs, showing a point-wise 95-percent confidence envelope around the fit (Figure 5):

```
> plot(mod.gam)
Press return for next page....
>
```

---

<sup>6</sup>The smoothing parameters are estimated along with the rest of the model, minimizing the generalized cross-validation criterion,

$$\frac{n\hat{\sigma}^2}{n - \text{df}_{\text{mod}}}$$

where  $\hat{\sigma}^2$  is the estimated error variance and  $\text{df}_{\text{mod}}$  is the equivalent degrees of freedom for the model, including both parametric and smooth terms. In the *generalized* additive model (considered below) the estimated dispersion  $\hat{\phi}$  replaces the error variance.

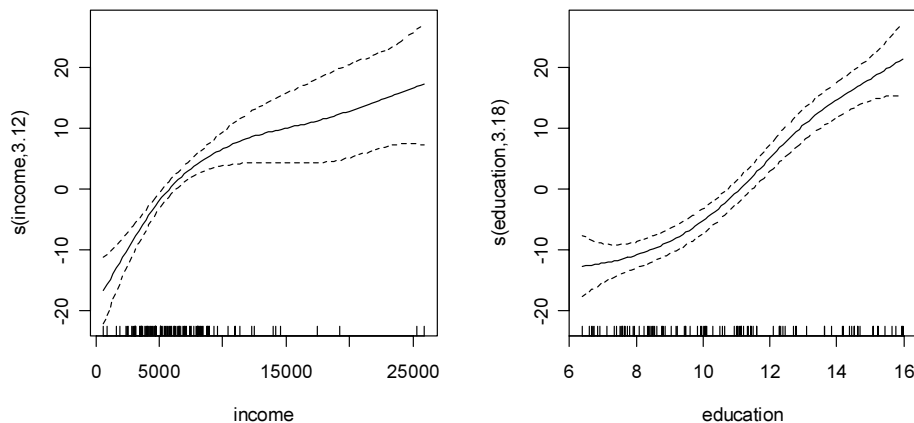


Figure 5: Partial-regression functions for the additive regression of `prestige` on `income` and `education`. The broken lines give point-wise 95-percent confidence envelopes around the fit.

The `gam` function is considerably more general than this example illustrates:

- The model can include smooth (interaction) terms in two or more predictors, for example, of the form `s(income, education)`.
- The model can be semi-parametric, including linear terms — for example, `prestige ~ s(income) + education`.
- Certain technical options, such as the kinds of splines employed, may be selected by the user.
- As its name implies (GAM = *generalized* additive model), the `gam` function is not restricted to models with normal errors and an identity link (see below).

### The `gam` Function in S-PLUS

Hastie and Tibshirani's (1990) `gam` function in S-PLUS differs from the version in the `mgcv` library in R: First, it is possible to fit partial-regression functions by local polynomial regression, using the `lo` function in a model formula, as well as by smoothing splines, using `s`. Second, the smoothing parameter for a term (the span for a local regression, or the degrees of freedom for a smoothing spline) is specified directly rather than determined by generalized cross-validation. As in R, the `gam` function in S-PLUS can also fit *generalized* additive models.

## 3 Generalized Nonparametric Regression

I will illustrate generalized nonparametric regression by fitting a logistic additive regression model to Mroz's labor-force participation data (described in Chapter 5). Recall that the response variable in this data set, `lfp`, is a factor coded `yes` for women in the labor-force, and `no` for those who are not. The predictors

include number of children five years of age or less (`k5`); number of children between the ages of six and 18 (`k618`); the woman's `age`, in years; factors indicating whether the woman (`wc`) and her husband (`hc`) attended college, `yes` or `no`; and family income (`inc`), excluding the wife's income and given in \$1000s. (I ignore the remaining variable in the data set, the log of the wife's expected wage rate, `lwg`; as explained in the text, the peculiar definition of `lwg` makes its use problematic.)

Because `k5` and `k618` are discrete, with relatively few distinct values, I will treat these predictors as factors, modeling them parametrically, along with the factors `wc` and `hc`:

```
> detach(Prestige)
> remove(list=objects()) # clean up everything
> data(Mroz)
> attach(Mroz)
> k5f <- factor(k5)
> k618f <- factor(k618)
> mod.1 <- gam(lfp ~ s(age) + s(inc) + k5f + k618f + wc + hc,
+   family=binomial)
```

```
> mod.1
```

```
Family: binomial
Link function: logit
```

```
Formula:
lfp ~ s(age) + s(inc) + k5f + k618f + wc + hc
```

```
Estimated degrees of freedom:
1.6605 1.8696 total = 17.53
```

```
GCV score: 1.0238
```

Printing the `gam` object shows only the degrees of freedom used for each smooth term (in the example, for `age` and `inc`), the degrees of freedom for the model as a whole (including both parametric and nonparametric terms), and the generalized cross-validation criterion (see footnote 6). Because there is no `summary` method for `gam` objects in the current version (0.6.2) of the `mgcv` library, I have written a simple method (which may be found in the script for this Appendix); applying the method to the example produces the following result:

```
> summary(mod.1)
Call:
gam(formula = lfp ~ s(age) + s(inc) + k5f + k618f + wc + hc,
    family = binomial)
```

```
Family: binomial
Link function: logit
```

```
Parametric Coefficients
```

	Estimate	Std. Error	z value	Pr(> z )
(Intercept)	0.537	0.181	2.97	0.0030
k5f1	-1.531	0.251	-6.10	1.1e-09
k5f2	-2.667	0.513	-5.20	2.0e-07
k5f3	-6.987	5.437	-1.28	0.1988
k618f1	-0.334	0.227	-1.47	0.1416
k618f2	-0.276	0.248	-1.12	0.2648
k618f3	-0.315	0.285	-1.10	0.2699
k618f4	-0.528	0.439	-1.20	0.2296

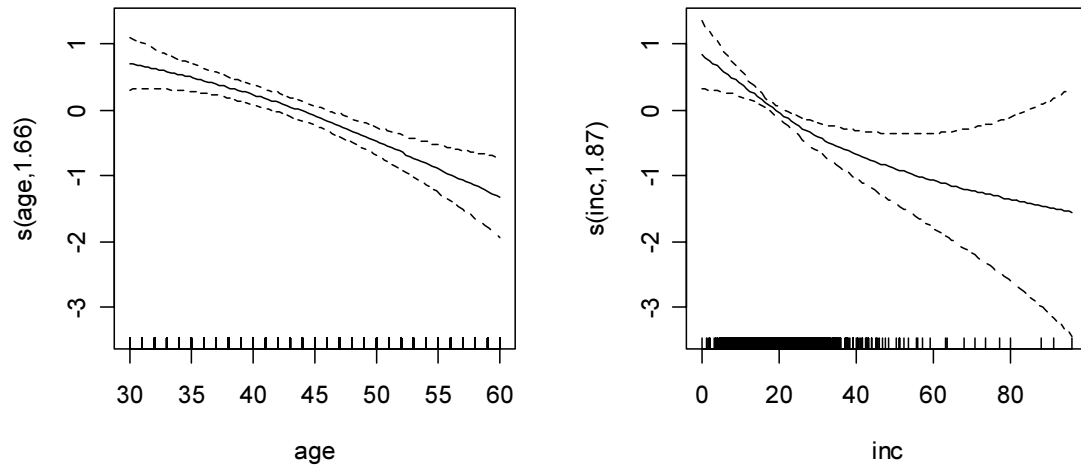


Figure 6: Smooth terms for `age` and `inc` in a semi-parametric generalized additive model for Mroz's labor-force participation data.

```

k618f5      -0.275      0.675     -0.41     0.6839
k618f6      -4.207      9.841     -0.43     0.6690
k618f7      -7.150      9.832     -0.73     0.4671
k618f8       4.510      9.831      0.46     0.6464
wcyes       0.987       0.223      4.42     9.9e-06
hcyes       0.158       0.207      0.77     0.4437

```

Degrees of freedom for terms in the model:

```

(Intercept)   s(age)   s(inc)   k5f   k618f
  1.0000     1.6605   1.8696   3.0000  8.0000
      wc      hc
  1.0000     1.0000

```

```

Null Deviance: 1029.7
Deviance: 911.32
GCV score: 1.0238
Residual Degrees of Freedom : 735.47
Dispersion: taken as 1
Number of Observations: 753

```

The `plot` method for `gam` objects graphs the smooth terms in the model, along with point-wise 95-percent confidence envelopes (Figure 6):

```

> plot(mod.1)
Press return for next page....
>

```

One use of additive regression models, including generalized additive models, is to test for nonlinearity: We may proceed by contrasting the deviance for a model that fits a term nonparametrically with the deviance

for an otherwise identical model that fits the term linearly. To illustrate, I replace the smooth term for `age` in the model with a linear term:

```
> mod.2 <- gam(lfp ~ age + s(inc) + k5f + k618f + wc + hc,
+ family=binomial)
> anova(mod.2, mod.1, test='Chisq')
Analysis of Deviance Table
```

```
Model 1: lfp ~ age + s(inc) + k5f + k618f + wc + hc
Model 2: lfp ~ s(age) + s(inc) + k5f + k618f + wc + hc
  Resid. Df Resid. Dev      Df Deviance P(>|Chi|)
1   736.177      913
2   735.470      911  0.707      2    0.092
```

Likewise, we can test for nonlinearity in the `inc` (income) effect:

```
> mod.3 <- gam(lfp ~ s(age) + inc + k5f + k618f + wc + hc,
+ family=binomial)
> anova(mod.3, mod.1, test='Chisq')
Analysis of Deviance Table
```

```
Model 1: lfp ~ s(age) + inc + k5f + k618f + wc + hc
Model 2: lfp ~ s(age) + s(inc) + k5f + k618f + wc + hc
  Resid. Df Resid. Dev      Df Deviance P(>|Chi|)
1   736.393      915
2   735.470      911  0.923      3    0.064
```

Neither test is quite statistically significant. (The `anova` method employed for these tests is not part of the `mgcv` library, but is also supplied in the script for the Appendix.)

Similarly, we can test the statistical significance of a term in the model by dropping it and noting the change in the deviance. For example, to test the `age` term:

```
> mod.4 <- gam(lfp ~ s(inc) + k5f + k618f + wc + hc, # omits age
+ family=binomial)
> anova(mod.4, mod.1, test='Chisq')
Analysis of Deviance Table
```

```
Model 1: lfp ~ s(inc) + k5f + k618f + wc + hc
Model 2: lfp ~ s(age) + s(inc) + k5f + k618f + wc + hc
  Resid. Df Resid. Dev      Df Deviance P(>|Chi|)
1     737.0      939
2     735.5      911  1.5      27  4.7e-07
```

Thus, the `age` effect is very highly statistically significant.

I leave it to the reader to perform similar tests for the other predictors in the model, including `inc` and the parametric terms.

## plot, summary, and anova Methods for GAMs in S-PLUS

The `plot` methods for GAMs in S-PLUS is more sophisticated than in the `mgcv` library in R. More information, such as partial residuals, may be included in the plot, and graphs can also be produced for parametric terms in the model. In addition, there are `summary` and `anova` methods for GAMs in S-PLUS.

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