

# Package ‘DPQmpfr’

January 20, 2025

**Title** DPQ (Density, Probability, Quantile) Distribution Computations  
using MPFR

**Version** 0.3-3

**VersionNote** Last CRAN: 0.3-2 on 2023-12-05; 0.3-1 on 2021-05-17

**Date** 2024-08-19

**Description** An extension to the 'DPQ' package with computations for 'DPQ'  
(Density (pdf), Probability (cdf) and Quantile) functions, where  
the functions here partly use the 'Rmpfr' package and hence the  
underlying 'MPFR' and 'GMP' C libraries.

**Depends** R (>= 3.6.0)

**Imports** DPQ (>= 0.5-3), Rmpfr (>= 0.9-0), gmp (>= 0.6-4), sfsmisc,  
stats, graphics, methods, utils

**Suggests** Matrix

**SuggestsNote** Matrix for its test-tools-1.R

**License** GPL (>= 2)

**Encoding** UTF-8

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**Author** Martin Maechler [aut, cre] (<<https://orcid.org/0000-0002-8685-9910>>)

**Maintainer** Martin Maechler <maechler@stat.math.ethz.ch>

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DPQmpfr-package	<i>DPQ (Density, Probability, Quantile) Distribution Computations using MPFR</i>
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## Description

An extension to the 'DPQ' package with computations for 'DPQ' (Density (pdf), Probability (cdf) and Quantile) functions, where the functions here partly use the 'Rmpfr' package and hence the underlying 'MPFR' and 'GMP' C libraries.

## Details

The DESCRIPTION file:

```

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Title:        DPQ (Density, Probability, Quantile) Distribution Computations using MPFR
Version:      0.3-3
VersionNote:  Last CRAN: 0.3-2 on 2023-12-05; 0.3-1 on 2021-05-17
Date:         2024-08-19
Authors@R:   person("Martin","Maechler", role=c("aut","cre"), email="maechler@stat.math.ethz.ch", comment = c(ORCID))
Description:  An extension to the 'DPQ' package with computations for 'DPQ' (Density (pdf), Probability (cdf) and Quantile) functions, where the functions here partly use the 'Rmpfr' package and hence the underlying 'MPFR' and 'GMP' C libraries.
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```

Maintainer: Martin Maechler <maechler@stat.math.ethz.ch>

#### Index of help topics:

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dbetaD94	Ding(1994) (non-central) Beta Distribution Functions
dhyperQ	Exact Hypergeometric Distribution Probabilites
dntJKBm	Non-central t-Distribution Density
gam1M	Compute $1/\Gamma(x+1) - 1$ Accurately
ldexp	Numeric / Mpfr Utilities for DPQmpfr
lgamma1pM	Compute $\log(\Gamma(x+1))$ Arbitrarily (MPFR) Accurately
pbeta_ser	Beta Distribution Function - 'BPSEr' Series Expansion from TOMS 708
pnormAsymp	Asymptotic Approximations of Extreme Tail 'pnorm()' and 'qnorm()'
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qbBaha2017	Accurate qbeta() values from Baharev et al (2017)'s Program
stirlerrM	Stirling Formula Approximation Error

#### Author(s)

Martin Maechler [aut, cre] (<<https://orcid.org/0000-0002-8685-9910>>)

Maintainer: Martin Maechler <maechler@stat.math.ethz.ch>

#### See Also

Packages **DPQ**, **Rmpfr** are both used by this package.

#### Examples

```
## An example how mpfr-numbers "just work" with reasonable R functions:
.srch <- search() ; doAtt <- is.na(match("Rmpfr:package", .srch))
if(doAtt) require(Rmpfr)
nu.s <- 2^seqMpfr(mpfr(-30, 64), mpfr(100, 64), by = 1/mpfr(4, 64))
b0 <- DPQ::b_chi(nu.s)
b1 <- DPQ::b_chi(nu.s, one.minus=TRUE)
stopifnot(inherits(b0,"mpfr"), inherits(b1, "mpfr"),
          b0+b1 == 1, diff(log(b1)) < 0)
plot(nu.s, log(b1), type="l", log="x")
plot(nu.s[-1], diff(log(b1)), type="l", log="x")
if(doAtt) # detach the package(s) we've attached above
  for(pkg in setdiff(search(), .srch)) detach(pkg, character.only=TRUE)
```

---

algdivM

---

Compute  $\log(\text{gamma}(b)/\text{gamma}(a+b))$  Accurately, also via **Rmpfr**


---

## Description

Computes

$$\text{algdiv}(a,b) := \log \frac{\Gamma(b)}{\Gamma(a+b)} = \log \Gamma(b) - \log \Gamma(a+b) = \text{lgamma}(b) - \text{lgamma}(a+b)$$

in a numerically stable way.

The name ‘algdiv’ is from the auxiliary function in R’s (TOMS 708) implementation of `pbeta()`. As package **DPQ** provides R’s Mathlib (double precision) as R function `algdiv()`, we append ‘M’ to show the reliance on the **Rmpfr** package.

## Usage

```
algdivM(a, b, usePr = NULL)
```

## Arguments

`a, b` numeric or numeric-alike vectors (recycled to the same length if needed), typically inheriting from class “mpfr”.

`usePr` positive integer specifying the precision in **bits**, or `NULL` when a smart default will be used.

## Details

Note that this is also useful to compute the Beta function

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

Clearly,

$$\log B(a,b) = \log \Gamma(a) + \text{algdiv}(a,b) = \log \Gamma(a) - \log Q_{ab}(a,b).$$

In our ‘`../tests/qbeta-dist.R`’ file, we look into computing  $\log(pB(p,q))$  accurately for  $p \ll q$ .

We are proposing a nice solution there.

How is this related to `algdiv()` ?

Additionally, we have defined

$$Q_{ab} = Q_{a,b} := \frac{\Gamma(a+b), \Gamma(b)}{\Gamma(a)},$$

such that  $\log Q_{ab}(a,b) := \log Q_{ab}(a,b)$  fulfills simply

$$\log Q_{ab}(a,b) = -\text{algdiv}(a,b)$$

see `logQab_asy` from package **DPQ**.

**Value**

a numeric vector of length  $\max(\text{length}(a), \text{length}(b))$  (if neither is of length 0, in which case the result has length 0 as well).

**Author(s)**

Martin Maechler (for the **Rmpfr** version).

**References**

Didonato, A. and Morris, A., Jr, (1992) Algorithm 708: Significant digit computation of the incomplete beta function ratios, *ACM Transactions on Mathematical Software* **18**, 360–373.

**See Also**

[gamma](#), [beta](#); the (double precision) version [algdiv\(\)](#) in **DPQ**, and also in **DPQ**, the asymptotic approximation [logQab\\_asy\(\)](#).

**Examples**

```
Qab <- algdivM(2:3, 8:14)
cbind(a = 2:3, b = 8:14, Qab) # recycling with a warning

## algdivM() and my logQab_asy() give *very* similar results for largish b:
(lQab <- DPQ::logQab_asy(3, 100))
all.equal( - algdivM(3, 100), lQab, tolerance=0) # 1.283e-16 !!
## relative error
1 + lQab/ algdivM(3, 1e10) # 0 (64b F 30 Linux; 2019-08-15)

## in-and outside of "certified" argument range {b >= 8}:
a. <- c(1:3, 4*(1:8))/32
b. <- seq(1/4, 20, by=1/4)
ad <- t(outer(a., b., algdivM))
## direct computation:
f.algdiv0 <- function(a,b) lgamma(b) - lgamma(a+b)
f.algdiv1 <- function(a,b) lgamma(b) - lgamma(a+b)
ad.d <- t(outer(a., b., f.algdiv0))

matplot(b., ad.d, type = "o", cex=3/4,
        main = quote(log(Gamma(b)/Gamma(a+b)) ~" vs. algdivM(a,b)"))
mtext(paste0("a[1:",length(a.),"] = ",
            paste0(paste(head(paste0(formatC(a.*32), "/32")), collapse=", "), ", .., 1")))
matlines(b., ad, type = "l", lwd=4, lty=1, col=adjustcolor(1:6, 1/2))
abline(v=1, lty=3, col="midnightblue")
# The larger 'b', the more accurate the direct formula wrt algdivM()
all.equal(ad[b. >= 1,], ad.d[b. >= 1,])# 1.5e-5
all.equal(ad[b. >= 2,], ad.d[b. >= 2,], tol=0)# 3.9e-9
all.equal(ad[b. >= 4,], ad.d[b. >= 4,], tol=0)# 4.6e-13
all.equal(ad[b. >= 6,], ad.d[b. >= 6,], tol=0)# 3.0e-15
all.equal(ad[b. >= 8,], ad.d[b. >= 8,], tol=0)# 2.5e-15 (not much better)
```

## Description

The three functions "p" (cumulative distribution, CDF), "d" (density (PDF)), and "q" (quantile) use Ding(1994)'s algorithm A, B, and C, respectively, each of which implements a recursion formula using only simple arithmetic and `log` and `exp`.

These are particularly useful also for using with high precision "mpfr" numbers from the **Rmpfr** CRAN package.

## Usage

```
dbetaD94(x, shape1, shape2, ncp = 0, log = FALSE,
         eps = 1e-10, itrmax = 100000L, verbose = FALSE)
pbetaD94(q, shape1, shape2, ncp = 0, lower.tail = TRUE, log.p = FALSE,
         log_scale = (a * b > 0) && (a + b > 100 || c >= 500),
         eps = 1e-10, itrmax = 100000L, verbose = FALSE)

qbetaD94(p, shape1, shape2, ncp = 0, lower.tail = TRUE, log.p = FALSE,
         log_scale = (a * b > 0) && (a + b > 100 || c >= 500),
         delta = 1e-6,
         eps = delta^2,
         itrmax = 100000L,
         iterN = 1000L,
         verbose = FALSE)
```

## Arguments

<code>x, q</code>	numeric vector of values in $[0, 1]$ as beta variates.
<code>shape1, shape2</code>	the two shape parameters of the beta distribution, must be positive.
<code>ncp</code>	the noncentrality parameter; by default zero for the ( <i>central</i> ) beta distribution; if positive, we have a noncentral beta distribution.
<code>p</code>	numeric vector of probabilities, <code>log()</code> ged in case <code>log.p</code> is true.
<code>log, log.p</code>	logical indicating if the density or probability values should be <code>log()</code> ged.
<code>lower.tail</code>	logical indicating if the lower or upper tail probability should be computed, or for <code>qbeta*()</code> are provided.
<code>eps</code>	a non-negative number specifying the desired accuracy for computing <code>F()</code> and <code>f()</code> .
<code>itrmax</code>	the maximal number of steps for computing <code>F()</code> and <code>f()</code> .
<code>delta</code>	[For <code>qbeta*()</code> :] non-negative number indicating the desired accuracy for computing $x_p$ (the root of $pbeta * () == p$ ), i.e., the convergence tolerance for the Newton iterations. This sets default <code>eps = delta^2</code> which is sensible but may be too small, such that <code>eps</code> should be specified in addition to <code>delta</code> .

iterN	[For qbeta*():] The maximal number of Newton iterations.
log_scale	logical indicating if most of the computations should happen in <code>log</code> scale, which protects from “early” overflow and underflow but takes more computations. The current default is somewhat <i>arbitrary</i> , still derived from the facts that <code>gamma(172)</code> overflows to <code>Inf</code> already and <code>exp(-750)</code> underflows to <code>0</code> already.
verbose	logical (or integer) indicating the amount of diagnostic output during computation; by default none.

### Value

In all three cases, a numeric vector with the same attributes as `x` (or `q` respectively), containing (an approximation) to the corresponding beta distribution function.

### Author(s)

Martin Maechler, notably `log_scale` was not part of Ding’s proposals.

### References

Cherng G. Ding (1994) On the computation of the noncentral beta distribution. *Computational Statistics & Data Analysis* **18**, 449–455.

### See Also

`pbeta`. Package `Rmpfr`’s `pbetaI()` needs both `shape1` and `shape2` to be integer but is typically more efficient than the current `pbetaD94()` implementation.

### Examples

```
## Low precision (eps, delta) values as "e.g." in Ding(94): -----

## Compare with Table 3 of Baharev_et_al 2017 % ==> ./qbBaha2017.Rd <<<<<<<<<<<<
aa <- c(0.5, 1, 1.5, 2, 2.5, 3, 5, 10, 25)
bb <- c(1:15, 10*c(2:5, 10, 25, 50))
utime <-
  qbet <- matrix(NA_real_, length(aa), length(bb),
                dimnames = list(a = formatC(aa), b = formatC(bb)))
(doExtras <- DPQmpfr::doExtras())
if(doExtras) qbetL <- utimeL <- utime

p <- 0.95
delta <- 1e-4
eps <- 1e-6
system.t.usr <- function(expr)
  system.time(gcFirst = FALSE, expr)[["user.self"]]

system.time(
  for(ia in seq_along(aa)) {
    a <- aa[ia]; cat("\n-----\na=", a, "\n")
    for(ib in seq_along(bb)) {
```

```

b <- bb[ib]; cat("\n>> b=",b,"\n")
utime [ia, ib] <- system.t.usr(
  qbet[ia, ib] <- qbetaD94(p, a, b, ncp = 0, delta=delta, eps=eps, verbose = 2))
if(doExtras)
  utimeL[ia, ib] <- system.t.usr(
    qbetL[ia, ib] <- qbetaD94(p, a, b, ncp = 0, delta=delta, eps=eps,
      verbose = 2, log_scale=TRUE))
  }
cat("\n")
}
)# system.time(.): ~ 1 sec (lynne i7-7700T, Fedora 32, 2020)
sum(print(table(round(1000*utime)))) # lynne .. :
## 0 1 2 3 4 5 6 7 8 9 10 11 14 15 16 29
## 53 94 15 3 3 12 2 2 2 2 1 2 3 1 2 1
## [1] 198
if(doExtras) print(sum(print(table(round(1000*utimeL)))) # lynne .. :

```

---

dhyperQ

*Exact Hypergeometric Distribution Probabilites*


---

### Description

Computes **exact** probabilities for the hypergeometric distribution (see, e.g., [dhyper\(\)](#) in R), using package **gmp**'s big integer and rational numbers, notably [chooseZ\(\)](#).

### Usage

```

dhyperQ(x, m, n, k)
phyperQ(x, m, n, k, lower.tail=TRUE)
phyperQall(m, n, k, lower.tail=TRUE)

```

### Arguments

x	the number of white balls drawn without replacement from an urn which contains both black and white balls.
m	the number of white balls in the urn.
n	the number of black balls in the urn.
k	the number of balls drawn from the urn, hence must be in $0, 1, \dots, m + n$ .
lower.tail	logical indicating if the lower or upper tail probability should be computed.

### Value

a bigrational (class "bigq" from package **gmp**) vector "as" x; currently of length one (as all the function arguments must be "scalar", currently).

### Author(s)

Martin Maechler



**See Also**

[chooseZ](#) (pkg **gmp**), and R's own [Hypergeometric](#)

**Examples**

```
## dhyperQ() is simply
function (x, m, n, k)
{
  stopifnot(k - x == as.integer(k - x))
  chooseZ(m, x) * chooseZ(n, k - x) / chooseZ(m + n, k)
}

# a case where phyper(11, 15, 0, 12, log=TRUE) gave 'NaN'
(phy5.0.12 <- cumsum(dhyperQ(0:12, m=15,n=0,k=12)))
stopifnot(phy5.0.12 == c(rep(0, 12), 1))

for(x in 0:9)
  stopifnot(phyperQ(x, 10,7,8) +
            phyperQ(x, 10,7,8, lower.tail=FALSE) == 1)

(ph. <- phyperQall(m=10, n=7, k=8))
## Big Rational ('bigq') object of length 8:
## [1] 1/2431 5/374 569/4862 2039/4862 3803/4862 4685/4862 4853/4862 1
stopifnot(identical(gmp::c_bigq(list(0, ph.)),
                  1- c(phyperQall(10,7,8, lower.tail=FALSE), 0)))

(doExtras <- DPQmpfr::doExtras())
if(doExtras) { # too slow for standard testing
  k <- 5000
  system.time(ph <- phyper(k, 2*k, 2*k, 2*k)) # 0 (< 0.001 sec)
  system.time(phQ <- phyperQ(k, 2*k, 2*k, 2*k)) # 5.6 (was 6.3) sec
  ## Relative error of R's phyper()
  stopifnot(print(gmp::asNumeric(1 - ph/phQ)) < 1e-14) # seen 1.063e-15
}
```

---

dnt

*Non-central t-Distribution Density*


---

**Description**

dntJKBm is a fully **Rmpfr**-ified vectorized version of [dntJKBf\(\)](#) from **DPQ**; see there.

dtWVm(*x*, *df*, *ncp*) computes the density function  $f(x)$  of the t distribution with *df* degrees of freedom and non-centrality parameter *ncp*, according to Wolfgang Viechtbauer's proposal in 2002, using an asymptotic formula for "large"  $df = \nu$ .

**Usage**

```
dntJKBm(x, df, ncp, log = FALSE, M = 1000) # __ Deprecated __ use DPQ :: dntJKBf
dtWVm (x, df, ncp, log = FALSE)
```

**Arguments**

x	numeric or "mpfr" vector.
df	degrees of freedom ( $> 0$ , maybe non-integer). $df = \text{Inf}$ is allowed.
ncp	non-centrality parameter $\delta$ ; If omitted, use the central t distribution.
log	as in <code>dt()</code> , a logical indicating if $\log(f(x, *))$ should be returned instead of $f(x, *)$ .
M	the number of terms to be used, a positive integer.

**Details**

See `dtWV`'s details (package **DPQ**).

As **DPQ**'s `dntJKBf()` is already fully mpfr-ized, `dntJKBm()` is deprecated.

**Value**

an mpfr vector of the same length as the maximum of the lengths of x, df, ncp.

**Note**

Package **DPQ**'s `dntJKBf()` is already fully mpfr-ized, and hence `dntJKBm()` is redundant, and therefore deprecated.

**Author(s)**

Martin Maechler

**See Also**

R's `dt`, and package **DPQ**'s `dntJKBf()` and `dtWV()`.

**Examples**

```
tt <- seq(0, 10, len = 21)
ncp <- seq(0, 6, len = 31)
dt3R <- outer(tt, ncp, dt, df = 3)
dt3WV <- outer(tt, ncp, dtWVm, df = 3)
all.equal(dt3R, dt3WV) # rel.err 0.00063
dt25R <- outer(tt, ncp, dt, df = 25)
dt25WV <- outer(tt, ncp, dtWVm, df = 25)
all.equal(dt25R, dt25WV) # rel.err 1.1e-5

x <- -10:700
fx <- dt(x, df = 22, ncp = 100)
lfx <- dt(x, df = 22, ncp = 100, log = TRUE)
lfv <- dtWVm(x, df = 22, ncp = 100, log = TRUE)

head(lfv, 15) # shows that R's dt(*, log=TRUE) implementation is "quite suboptimal"

## graphics
```

```

opa <- par(no.readonly=TRUE)
par(mar=.1+c(5,4,4,3), mgp = c(2, .8,0))
plot(fx ~ x, type="l")
par(new=TRUE) ; cc <- c("red", adjustcolor("orange", 0.4))
plot(lfx ~ x, type = "o", pch=".", col=cc[1], cex=2, ann=FALSE, yaxt="n")
sfsmisc::eaxis(4, col=cc[1], col.axis=cc[1], small.args = list(col=cc[1]))
lines(x, lfV, col=cc[2], lwd=3)
dtt1 <- "      dt"; dtt2 <- "(x, df=22, ncp=100"; dttL <- paste0(dtt2,", log=TRUE)")
legend("right", c(paste0(dtt1,dtt2,"")), paste0(c(dtt1,"dtWVm"), dttL)),
      lty=1, lwd=c(1,1,3), col=c("black", cc), bty = "n")
par(opa) # reset

## For dntJKBm(), see  example(dntJKBf, package="DPQ")

```

---

DPQmpfr-utils

*Numeric / Mpfr Utilities for DPQmpfr*


---

## Description

Utilities for package **DPQmpfr**

## Usage

```
ldexp(f, E)
```

## Arguments

f                   ‘fraction’, as such with absolute value in  $[0.5, 1)$ , but can be any numbers.  
E                   integer-valued exponent(s).

## Details

ldexp() is a simple wrapper, either calling DPQ: : `ldexp` from **DPQ** or `ldexpMpfr` from the **Rmpfr** package,

$$ldexp(f, E) := f \times 2^E,$$

computed accurately and fast on typical platforms with internally binary arithmetic.

## Value

either a numeric or a "mpfr", depending on the type of f, vector as (the recycled) combination of f and E.

## See Also

`ldexp` from package **DPQ** and `ldexpMpfr` from package **Rmpfr**.

**Examples**

```
ldexp(1:10, 2)
ldexp(Rmpfr::Const("pi", 96), -2:2) # = pi * (1/4 1/2 1 2 4)
```

---

gam1M	<i>Compute 1/Gamma(x+1) - 1 Accurately</i>
-------	--

---

**Description**

FIXME: "R's own" double prec version is now in package DPQ: e.g. `~/R/Pkgs/DPQ/man/gam1.Rd`

FIXME2: R-only implementation is in `~/R/Pkgs/DPQ/TODO_R_versions_gam1_etc.R`

Computes  $1/\Gamma(a+1) - 1$  accurately in  $[-0.5, 1.5]$  for numeric argument  $a$ ; For "mpfr" numbers, the precision is increased intermediately such that  $a+1$  should not lose precision.

**Usage**

```
gam1M(a, usePr = NULL)
```

**Arguments**

**a** a numeric or numeric-alike, typically inheriting from class "mpfr".

**usePr** the precision to use; the default, NULL, means to use a default which depends on **a**, specifically `getPrec(a)`.

**Details**

<https://dlmf.nist.gov/> states the well-know Taylor series for

$$\frac{1}{\Gamma(z)} = \sum_{k=1}^{\infty} c_k z^k$$

with  $c_1 = 1$ ,  $c_2 = \gamma$ , (Euler's gamma,  $\gamma = 0.5772\dots$ ), with recursion  $c_k = (\gamma c_{k-1} - \zeta(2) c_{k-2} \dots + (-1)^k \zeta(k-1) c_1) / (k-1)$ .

Hence,

$$\frac{1}{\Gamma(z+1)} = z + 1 + \sum_{k=2}^{\infty} c_k (z+1)^k,$$

$$\frac{1}{\Gamma(z+1)} - 1 = z + \gamma * (z+1)^2 + \sum_{k=3}^{\infty} c_k (z+1)^k.$$

Consequently, for  $\zeta_k := \zeta(k)$ ,  $c_3 = (\gamma^2 - \zeta_2)/2$ ,  $c_4 = \gamma^3/6 - \gamma\zeta_2/2 + \zeta_3/3$ .

```

require(Rmpfr) # Const(), mpfr(), zeta()
gam <- Const("gamma", 128)
z <- zeta(mpfr(1:7, 128))
(c3 <- (gam^2 - z[2])/2) # -0.655878071520253881077019515145
(c4 <- (gam*c3 - z[2]*c2 + z[3])/3) # -0.04200263503409523552900393488
(c4 <- gam*(gam^2/6 - z[2]/2) + z[3]/3)
(c5 <- (gam*c4 - z[2]*c3 + z[3]*c2 - z[4])/4) # 0.1665386113822914895017007951
(c5 <- (gam^4/6 - gam^2*z[2] + z[2]^2/2 + gam*z[3]*4/3 - z[4])/4)

```

**Value**

a numeric-alike vector like a.

**Author(s)**

Martin Maechler building on C code of TOMS 708

**References**

TOMS 708, see [pbeta](#)

**See Also**

[gamma](#).

**Examples**

```

##' naive direct formula:
g1 <- function(u) 1/gamma(u+1) - 1

##' @title gam1() from TOMS 708 -- translated to R (*and* vectorized)
##' @author Martin Maechler
gam1R <- function(a, chk=TRUE) { ## == 1/gamma(a+1) - 1 -- accurately ONLY for -0.5 <= a <= 1.5
  if(!length(a)) return(a)
  ## otherwise:
  if(chk) stopifnot(-0.5 <= a, a <= 1.5) # if not, the computation below is non-sense!
  d <- a - 0.5
  ## t := if(a > 1/2) a-1 else a ==> t in [-0.5, 0.5] <==> |t| <= 0.5
  R <- t <- a
  dP <- d > 0
  t[dP] <- d[dP] - 0.5
  if(any(N <- (t < 0.))) { ## L30: */
    r <- c(-.422784335098468, -.771330383816272,
           -.244757765222226, .118378989872749, 9.30357293360349e-4,
           -.0118290993445146, .00223047661158249, 2.66505979058923e-4,
           -1.32674909766242e-4)
    s1 <- .273076135303957
    s2 <- .0559398236957378
    t_ <- t[N]
    top <- ((((((r[9] * t_ + r[8]) * t_ + r[7]) * t_ + r[6]) * t_ + r[5]) * t_ + r[4]
    ) * t_ + r[3]) * t_ + r[2]) * t_ + r[1]
  }
}

```

```

    bot <- (s2 * t_ + s1) * t_ + 1.
    w <- top / bot
    ## if (d > 0.) :
    if(length(iP <- which(dP[N])))
      R[N & dP] <- (t_ * w)[iP] / a[N & dP]
    ## else d <= 0 :
    if(length(iN <- which(!dP[N])))
      R[N & !dP] <- a[N & !dP] * (w[iN] + 0.5 + 0.5)
  }
  if(any(Z <- (t == 0))) ## L10: a in {0, 1}
    R[Z] <- 0.
  if(any(P <- t > 0)) { ## t > 0; L20: */
    p <- c( .577215664901533, -.409078193005776,
           -.230975380857675, .0597275330452234, .0076696818164949,
           -.00514889771323592, 5.89597428611429e-4 )
    q <- c(1., .427569613095214, .158451672430138, .0261132021441447, .00423244297896961)
    t <- t[P]
    top <- (((((p[7] * t + p[6])*t + p[5])*t + p[4])*t + p[3])*t + p[2])*t + p[1]
    bot <- (((q[5] * t + q[4]) * t + q[3]) * t + q[2]) * t + 1.
    w <- top / bot
    ## if (d > 0.) ## L21: */
    if(length(iP <- which(dP[P])))
      R[P & dP] <- t[iP] / a[P & dP] * (w[iP] - 0.5 - 0.5)
    ## else d <= 0 :
    if(length(iN <- which(!dP[P])))
      R[P & !dP] <- a[P & !dP] * w[iN]
  }
}
R
} ## gam1R()

u <- seq(-.5, 1.5, by=1/16); set.seed(1); u <- sample(u) # permuted (to check logic)
g11 <- vapply(u, gam1R, 1) # [-.5, 1.5] == the interval for which the above gam1() was made
gam1. <- gam1R(u)
cbind(u, gam1., D = sfsmisc::relErrV(gam1., g1(u)))[order(u),]
# looks "too good", as we are not close (but different) to {0, 1}
stopifnot( identical(g11, gam1.) )
all.equal(g1(u), gam1., tolerance = 0) # 6.7e-16 ("too good", see above)
stopifnot( all.equal(g1(u), gam1.) )

## Comparison using Rmpfr; slightly extending [-.5, 1.5] interval (and getting much closer to {0,1})
u <- seq(-0.525, 1.525, length.out = 2001)
uM <- Rmpfr::mpfr(u, 128)
gam1M. <- gam1M(uM)
relE <- Rmpfr::asNumeric(sfsmisc::relErrV(gam1M., gam1R(u, chk=FALSE)))
rbind(rErr = summary(relE),
      `|rE|` = summary(abs(relE)))
##           Min.   1st Qu.   Median     Mean   3rd Qu.     Max.
## rErr -3.280e-15 -3.466e-16 1.869e-17 1.526e-16 4.282e-16 1.96e-14
## |rE|  1.343e-19  2.363e-16  3.861e-16  6.014e-16  6.372e-16  1.96e-14
stopifnot(max(abs(relE)) < 1e-13)

relEtit <- expression("Relative Error of " ~ gam1(u) %~%{} == frac(1, Gamma(u+1)) - 1) #%
plot(relE ~ u, type="l", ylim = c(-1,1) * 2.5e-15, main = relEtit)

```

```

grid(lty = 3); abline(v = c(-.5, 1.5), col = adjustcolor(4, 1/2), lty=2, lwd=2)

## what about the direct formula -- how bad is it really ?
relED <- Rmpfr::asNumeric(sfsmisc::relErrV(gam1M., g1(u)))

plot(relE ~ u, type="l", ylim = c(-1,1) * 1e-14, main = relEtit)
lines(relED ~ u, col = adjustcolor(2, 1/2), lwd = 2); abline(v = (-1:3)/2, lty=2, col="orange3")
mtext("comparing with direct formula 1/gamma(u+1) - 1", col=2, cex=3/4)
legend("top", c("gam1R(u)", "1/gamma(u+1) - 1"), col = 1:2, lwd=1:2, bty="n")
## direct is clearly *worse* , but not catastrophic

```

---

lgamma1pM

*Compute  $\log(\Gamma(x+1))$  Arbitrarily (MPFR) Accurately*


---

### Description

Computes  $\log \Gamma(x + 1)$  accurately notably when  $|x| \ll 1$ . For "mpfr" numbers, the precision is increased intermediately such that  $a + 1$  should not lose precision.

R's "own" double prec version is soon available in package in **DPQ**, under the name `gamln1()` (from TOMS 708).

### Usage

```
lgamma1pM(a, usePr = NULL, DPQmethod = c("lgamma1p", "alгам1"))
```

### Arguments

<code>a</code>	a numeric or numeric-alike vector, typically inheriting from class "mpfr".
<code>usePr</code>	positive integer specifying the precision in <b>bits</b> , or <b>NULL</b> when a smart default will be used.
<code>DPQmethod</code>	a character string; must be the name of an <code>lgamma1p()</code> -alike function from package <b>DPQ</b> . It will be called in case of <code>is.numeric(a)</code> (and when <b>DPQ</b> is available).

### Value

a numeric-alike vector like `a`.

### Author(s)

Martin Maechler

### References

TOMS 708, see [pbeta](#)

**See Also**

`lgamma()` (and `gamma()` (same page)), and our `algdivM()`; further, package **DPQ**'s `lgamma1p()` and (if already available) `gamln1()`.

**Examples**

```
## Package {DPQ}'s lgamma1p():
lgamma1p <- DPQ::lgamma1p
lg1 <- function(u) lgamma(u+1) # the simple direct form
u <- seq(-.5, 1.5, by=1/16); set.seed(1); u <- sample(u) # permuted (to check logic)
g11 <- vapply(u, lgamma1p, numeric(1))
lgamma1p. <- lgamma1p(u)
all.equal(lg1(u), g11, tolerance = 0) # see 3.148e-16
stopifnot(exprs = {
  all.equal(lg1(u), g11, tolerance = 2e-15)
  identical(g11, lgamma1p.)
})

## Comparison using Rmpfr; slightly extending the [-.5, 1.5] interval:
u <- seq(-0.525, 1.525, length.out = 2001)
lg1p <- lgamma1pM( u)
lg1pM <- lgamma1pM(Rmpfr::mpfr(u, 128))
asNumeric <- Rmpfr::asNumeric
relErrV <- sfsmisc::relErrV
if(FALSE) { # DPQ "latest" version __FIXME__
  lng1 <- DPQ::lngam1(u)
  relE <- asNumeric(relErrV(lg1pM, cbind(lgamma1p = lg1p, lngam1 = lng1)))
} else {
  relE <- asNumeric(relErrV(lg1pM, cbind(lgamma1p = lg1p)))# , lngam1 = lng1)))
}

## FIXME: lgamma1p() is *NOT* good around u=1. -- even though it should
## and the R-only vs (not installed) *does* "work" (is accurate there) ?????
## --> ~/R/Pkgs/DPQ/TODO_R_versions_gam1_etc.R
if(FALSE) {
  matplot(u, relE, type="l", ylim = c(-1,1) * 2.5e-15,
    main = expression("relative error of " ~~ lgamma1p(u) == log( Gamma(u+1) )))
} else {
  plot(relE ~ u, type="l", ylim = c(-1,1) * 2.5e-15,
    main = expression("relative error of " ~~ lgamma1p(u) == log( Gamma(u+1) )))
}
grid(lty = 3); abline(v = c(-.5, 1.5), col = adjustcolor(4, 1/2), lty=2, lwd=2)

## what about the direct formula -- how bad is it really ?
relED <- asNumeric(relErrV(lg1pM, lg1(u)))
lines(relED ~ u, col = adjustcolor(2, 1/2), lwd = 2)
```



## Description

Compute a version of the Beta cumulative distribution function (`pbeta()` in R), namely using the series expansion, named `BPSER()`, from “TOMS 708”, i.e., Didonato and Morris (1992).

This “pure R” function exists for didactical or documentational reasons on one hand, as R’s own `pbeta()` uses this expansion when appropriate and other algorithms otherwise. On the other hand, using high precision `q` and MPFR arithmetic (via package `Rmpfr`) may allow to get highly accurate `pbeta()` values.

## Usage

```
pbeta_ser(q, shape1, shape2, log.p = FALSE, eps = 1e-15, errPb = 0, verbose = FALSE)
```

## Arguments

<code>q</code> , <code>shape1</code> , <code>shape2</code>	quantiles and shape parameters of the Beta distribution, <code>q</code> typically in $[0, 1]$ , see <code>pbeta</code> . Here, <code>q</code> <i>must be scalar</i> , i.e., of length one, and may inherit from class “ <code>mpfr</code> ”, in order to be more accurate (than with the double precision computations).
<code>log.p</code>	if TRUE, probabilities <code>p</code> are given as $\log(p)$ .
<code>eps</code>	non-negative number; <code>tol &lt;- eps/shape1</code> will be used for convergence checks in the series computations.
<code>errPb</code>	an integer code, typically in $-2, -1, 0$ to determine how warnings on convergence failures are handled.
<code>verbose</code>	logical indicating if console output about intermediate results should be printed.

## Details

`pbeta_ser()` crucially needs three auxiliary functions which we “mpfr-ized” as well: `gam1M()`, `lgamma1pM()`, and `algdivM`.

## Value

An approximation to the Beta probability  $P[X \leq q]$  for  $X \sim B(a, b)$ , (where  $a = \text{shape1}$ , and  $b = \text{shape2}$ ).

## Author(s)

Didonato and Morris and R Core team; separate packaging by Martin Maechler.

## References

Didonato, A. and Morris, A., Jr, (1992) Algorithm 708: Significant digit computation of the incomplete beta function ratios, *ACM Transactions on Mathematical Software* **18**, 360–373; doi:[10.1145/131766.131776](https://doi.org/10.1145/131766.131776).

**See Also**

`pbeta`, **DPQmpfr**'s own `pbetaD94`; even more `pbeta()` approximations in package **DPQ**, e.g., `pnbetaAS310`, or `pbetaRv1`.

In addition, for integer shape parameters, the potentially “fully accurate” finite sum base `pbetaI()` in package **Rmpfr**.

**Examples**

```
(p. <- pbeta_ser(1/2, shape1 = 2, shape2 = 3, verbose=TRUE))
(lp <- pbeta_ser(1/2, shape1 = 2, shape2 = 3, log.p = TRUE))
  all.equal(lp, log(p.), tolerance=0) # 1.48e-16
stopifnot(all.equal(lp, log(p.), tolerance = 1e-13))

## Using Vectorize() in order to allow vector 'q' e.g. for curve():
str(pbetaSer <- Vectorize(pbeta_ser, "q"))
curve(pbetaSer(x, 1.5, 4.5)); abline(h=0:1, v=0:1, lty=2, col="gray")
curve(pbeta(x, 1.5, 4.5), add=TRUE, col = adjustcolor(2, 1/4), lwd=3)

## now using mpfr-numbers:
half <- 1/Rmpfr::mpfr(2, 256)
(p2 <- pbeta_ser(half, shape1 = 1, shape2 = 123))
```

---

pnormLU

*Bounds for  $1 - \Phi(x)$  – Mill's Ratio related Bounds for  $pnorm()$*

---

**Description**

Bounds for  $1 - \Phi(x)$ , i.e., `pnorm(x, *, lower.tail=FALSE)`, typically related to Mill's Ratio.

**Usage**

```
pnormL_LD10(x, lower.tail = FALSE, log.p = FALSE)
pnormU_S53(x, lower.tail = FALSE, log.p = FALSE)
```

**Arguments**

`x` positive (at least non-negative) numeric "`mpfr`" vector (or `array`).  
`lower.tail, log.p` logical, see, e.g., `pnorm()`.

**Value**

vector/array/mpfr like `x`.

**Author(s)**

Martin Maechler

## References

Lutz Duembgen (2010) *Bounding Standard Gaussian Tail Probabilities*; arXiv preprint 1012.2063, <https://arxiv.org/abs/1012.2063>

## See Also

`pnorm`. The same functions “numeric-only” are in my **DPQ** package.

## Examples

```
x <- seq(1/64, 10, by=1/64)
px <- cbind(
  lQ = pnorm      (x, lower.tail=FALSE, log.p=TRUE)
  , Lo = pnormL_LD10(x, lower.tail=FALSE, log.p=TRUE)
  , Up = pnormU_S53 (x, lower.tail=FALSE, log.p=TRUE))
matplot(x, px, type="l") # all on top of each other

matplot(x, (D <- px[,2:3] - px[,1]), type="l") # the differences
abline(h=0, lty=3, col=adjustcolor(1, 1/2))

## check they are lower and upper bounds indeed :
stopifnot(D[, "Lo"] < 0, D[, "Up"] > 0)

matplot(x[x>4], D[x>4,], type="l") # the differences
abline(h=0, lty=3, col=adjustcolor(1, 1/2))

### zoom out to larger x : [1, 1000]
x <- seq(1, 1000, by=1/4)
px <- cbind(
  lQ = pnorm      (x, lower.tail=FALSE, log.p=TRUE)
  , Lo = pnormL_LD10(x, lower.tail=FALSE, log.p=TRUE)
  , Up = pnormU_S53 (x, lower.tail=FALSE, log.p=TRUE))
matplot(x, px, type="l") # all on top of each other
matplot(x, (D <- px[,2:3] - px[,1]), type="l") # the differences
abline(h=0, lty=3, col=adjustcolor(1, 1/2))

## check they are lower and upper bounds indeed :
table(D[, "Lo"] < 0) # no longer always true
table(D[, "Up"] > 0)
## not even when equality (where it's much better though):
table(D[, "Lo"] <= 0)
table(D[, "Up"] >= 0)

## *relative* differences:
matplot(x, (rD <- 1 - px[,2:3] / px[,1]), type="l", log = "x")
abline(h=0, lty=3, col=adjustcolor(1, 1/2))
## abs()
matplot(x, abs(rD), type="l", log = "xy", axes=FALSE, # NB: curves *cross*
  main = "relative differences 1 - pnormUL(x, *) / pnorm(x, *)")
legend("top", c("Low.Bnd(D10)", "Upp.Bnd(S53)"), bty="n", col=1:2, lty=1:2)
sfsmisc::eaxis(1, sub10 = 2)
sfsmisc::eaxis(2)
```

```

abline(h=(1:4)*2^-53, col=adjustcolor(1, 1/4))

### zoom out to LARGE x : -----

x <- 2^seq(0, 30, by = 1/64)
col4 <- adjustcolor(1:4, 1/2)
options(width = 111) -> oop # (nicely printing "tables")
if(FALSE)## or even HUGE:
  x <- 2^seq(4, 513, by = 1/16)
px <- cbind(
  lQ = pnorm(x, lower.tail=FALSE, log.p=TRUE)
  , a0 = dnorm(x, log=TRUE)
  , a1 = dnorm(x, log=TRUE) - log(x)
  , Lo = pnormL_LD10(x, lower.tail=FALSE, log.p=TRUE)
  , Up = pnormU_S53(x, lower.tail=FALSE, log.p=TRUE))
doLegTit <- function(col=1:4) {
  title(main = "relative differences 1 - pnormUL(x, *)/pnorm(x,*)")
  legend("top", c("phi(x)", "phi(x)/x", "Low.Bnd(D10)", "Upp.Bnd(S53)"),
        bty="n", col=col, lty=1:4)
}
## *relative* differences are relevant:
matplot(x, (rD <- 1 - px[,-1] / px[,1]), type="l", log = "x",
        ylim = c(-1,1)/2^8, col=col4) ; doLegTit()
abline(h=0, lty=3, col=adjustcolor(1, 1/2))

if(x[length(x)] > 1e150) # the "HUGE" case (not default)
  print( tail(cbind(x, px), 20) )
##--> For very large x ~ 1e154, the approximations overflow *later* than pnorm() itself !!

## abs(rel.Diff) ---> can use log-log:
matplot(x, abs(rD), type="l", log = "xy", xaxt="n", yaxt="n"); doLegTit()
sfsmisc::eaxis(1, sub10=2)
sfsmisc::eaxis(2)
abline(h=(1:4)*2^-53, col=adjustcolor(1, 1/4))

## lower.tail=TRUE (w/ log.p=TRUE) works "the same" for x < 0:
require(Rmpfr)
x <- - 2^seq(0, 30, by = 1/64)
## ==
log1mexp <- Rmpfr::log1mexp # Rmpfr version >= 0.8-2 (2020-11-11 on CRAN)
px <- cbind(
  lQ = pnorm(x, lower.tail=TRUE, log.p=TRUE)
  , a0 = log1mexp(- dnorm(-x, log=TRUE))
  , a1 = log1mexp(-(dnorm(-x, log=TRUE) - log(-x)))
  , Lo = log1mexp(-pnormL_LD10(-x, lower.tail=TRUE, log.p=TRUE))
  , Up = log1mexp(-pnormU_S53(-x, lower.tail=TRUE, log.p=TRUE)) )
matplot(-x, (rD <- 1 - px[,-1] / px[,1]), type="l", log = "x",
        ylim = c(-1,1)/2^8, col=col4) ; doLegTit()
abline(h=0, lty=3, col=adjustcolor(1, 1/2))

## Comparison with Rmpfr::erf() / erfc() based pnorm():

```

```

## Set the exponential ranges to maximal -- to evade underflow as long as possible
.mpfr_erange_set(value = (1-2^-52) * .mpfr_erange(c("min.emin","max.emax")))
l2t <- seq(0, 32, by=1/4)
twos <- mpfr(2, 1024)^l2t
Qt <- pnorm(twos, lower.tail=FALSE)
pnU <- pnormU_S53 (twos, log.p=TRUE)
pnL <- pnormL_LD10(twos, log.p=TRUE)
logQt <- log(Qt)
M <- cbind(twos, Qt, logQt = logQt, pnU)
roundMpfr(M, 40)
dM <- asNumeric(cbind(dU = pnU - logQt, dL = logQt - pnL,
# NB: the numbers are *negative*
rdU= 1 - pnU/logQt, rdL = pnL/logQt - 1))
data.frame(l2t, dM)
## The bounds are ok (where Qt does not underflow): L < p < U :
stopifnot(pnU > pnL, pnU > logQt, (logQt > pnL)[Qt > 0])
roundMpfr(cbind(twos, pnL, pnU, D=pnU-pnL, relD=(pnU-pnL)/((pnU+pnL)/2)), 40)

## ----- R's pnorm() -- is it always inside [L, U] ?? -----
nQt <- stats::pnorm(asNumeric(twos), lower.tail=FALSE, log.p=TRUE)
data.frame(l2t, check.names=FALSE
, nQt
, "L <= p" = c(" ", "W")[2 -(pnL <= nQt)]
, "p <= U" = c(" ", "W")[2- (nQt <= pnU)])
## ==> pnorm() is *outside* sometimes for l2t >= 7.25; always as soon as l2t >= 9.25

## *but* the relative errors are around c_epsilon in all these cases :
plot (2^l2t, asNumeric(abs(nQt-pnL)/abs(pnU)), type="o", cex=1/4, log="xy", axes=FALSE)
sfsmisc::eaxis(1, sub10 = 2)
sfsmisc::eaxis(2)
lines(2^l2t, asNumeric(abs(nQt-pnU)/abs(pnU)), type="o", cex=1/4, col=2)
abline(h=c(1:4)*2^-53, lty=2, col=adjustcolor(1, 1/4))

options(oop)# reverting

```

---

pqnormAsymp

*Asymptotic Approximations of Extreme Tail 'pnorm()' and 'qnorm()'*

---

## Description

These functions provide the first terms of asymptotic series approximations to `pnorm()`'s (extreme) tail, from Abramowitz and Stegun's 26.2.13 (p.932), or `qnorm()` where the approximations have been derived via iterative plugin using Abramowitz and Stegun's formula.

## Usage

```

pnormAsymp(x, k, lower.tail = FALSE, log.p = FALSE)
qnormAsymp(p, lp = .DT_Clog(p, lower.tail = lower.tail, log.p = log.p),
order, M_2PI =,
lower.tail = TRUE, log.p = missing(p))

```

**Arguments**

x	positive (at least non-negative) numeric vector.
k	integer $\geq 0$ indicating how many terms the approximation should use; currently $k \leq 5$ .
p	numeric vector of probabilities, possibly transformed, depending on log.p. Does not need to be specified, if lp is instead.
lp	numeric (vector) of $\log(1-p)$ values; if not specified, computed from p, depending on lower.tail and log.p.
order	an integer in $\{0, 1, \dots, 5\}$ , specifying the approximation order.
M_2PI	the number $2\pi$ in the same precision as p or lp, i.e., <code>numeric</code> or of class <code>"mpfr"</code> .
lower.tail	logical; if true, probabilities are $P[X \leq x]$ , otherwise upper tail probabilities, $P[X > x]$ .
log.p	logical; if TRUE (default for qnormAsymp !!), probabilities $p$ are given as $\log(p)$ in argument p or $\log(1-p)$ in lp.

**Details**

see *both* help pages [pnormAsymp](#) and [qnormAsymp](#) from our package **DPQ**.

**Value**

vector/array/mpfr like first argument x or p or lp, respectively.

**Author(s)**

Martin Maechler

**See Also**

[pnorm](#). The same functions “numeric-only” are in my **DPQ** package with more extensive documentation.

**Examples**

```
require("Rmpfr") # (in strong dependencies of this pkg {DPQmpfr})
x <- seq(1/64, 10, by=1/64)
xm <- mpfr(x, 96)
"TODO"

## More extreme tails: -----
##
## 1. pnormAsymp() -----
lx <- c((2:10)*2, 25, (3:9)*10, (1:9)*100, (1:8)*1000, (2:7)*5000)
lxm <- mpfr(lx, 256)
Px <- pnorm(lxm, lower.tail = FALSE, log.p=TRUE)
PxA <- sapplyMpfpr(setNames(0:5, paste("k =", 0:5)),
  pnormAsymp, x=lxm, lower.tail = FALSE, log.p=TRUE)
if(interactive())
```

```

    roundMpfr(PxA, 40)
# rel.errors :
relE <- asNumeric(1 - PxA/Px)
options(width = 99) -> oop # (nicely printing the matrices)
cbind(lx, relE)
matplot(lx, abs(relE), type="b", cex = 1/2, log="xy", pch=as.character(0:5),
        axes=FALSE,
        main = "|relE( <pnormAsymp(lx, k=*, lower.tail=FALSE, log.p=TRUE) )|")
sfsmisc::eaxis(1, sub10=2); sfsmisc::eaxis(2)
legend("bottom", paste("k =", 0:5), col=1:6, lty=1:5,
      pch = as.character(0:5), pt.cex=1/2, bty="n")
## NB: rel.Errors go down to 7e-59 ==> need precision of -log2(7e-59) ~ 193.2 bits

## 2. qnormAsymp() -----
QPx <- sapplyMpfr(setNames(0:5, paste("k =", 0:5)),
                 function(k) qnormAsymp(Px, order=k, lower.tail = FALSE, log.p=TRUE))
(relE.q <- asNumeric(QPx/lx - 1))
    # note how consistent the signs are (!) <==> have upper/lower bounds

matplot(-asNumeric(Px), abs(relE.q), type="b", cex = 1/2, log="xy", pch=as.character(0:5),
        xlab = quote(-Px), axes=FALSE,
        main = "|relE( <qnormAsymp(Px, k=*, lower.tail=FALSE, log.p=TRUE) )|")
sfsmisc::eaxis(1, sub10=2); sfsmisc::eaxis(2)
legend("bottom", paste("k =", 0:5), col=1:6, lty=1:5,
      pch = as.character(0:5), pt.cex=1/2, bty="n")

options(oop) # {revert to previous state}

```

## Description

Compute "accurate" `qbeta()` values from Baharev et al (2017)'s Program.

## Usage

```
data("qbBaha2017")
```

## Format

FIXME: Their published table only shows 6 digits, but running their (32-bit statically linked) Linux executable 'mindiffver' (from their github repos, see "source") with their own 'input.txt' gives 12 digits accuracy, which we should be able to increase even more, see <https://github.com/baharev/mindiffver/blob/master/README.md>

A numeric matrix,  $9 \times 22$  with guaranteed accuracy `qbeta(0.95, a, b)` values, for  $a = 0.5, 1, 1.5, 2, 2.5, 3, 5, 10, 25$  and  $b =$  with `str()`

```

num [1:9, 1:22] 0.902 0.95 0.966 0.975 0.98 ...
- attr(*, "dimnames")=List of 2
..$ a: chr [1:9] "0.5" "1" "1.5" "2" ...
..$ b: chr [1:22] "1" "2" "3" "4" ...

```

## Details

MM constructed this data as follows (TODO: say more..):

```

ff <- "~/R/MM/NUMERICS/dpq-functions/beta-gamma-etc/Baharev_et_al-2017_table3.txt"
qbB2017 <- t( data.matrix(read.table(ff)) )
dimnames(qbB2017) <- dimnames(qbet)
saveRDS(qbB2017, "....../qbBaha2017.rds")

```

## Source

This matrix comprises all entries of Table 3, p. 776 of Baharev, A., Schichl, H. and Rév, E. (2017) Computing the noncentral-F distribution and the power of the F-test with guaranteed accuracy; *Comput. Stat.* **32**(2), 763–779. doi:10.1007/s00180016-07013

The paper mentions the first author's 'github' repos where source code and executables are available from: <https://github.com/baharev/mindiffver/>

## Examples

```

data(qbBaha2017)
str(qbBaha2017)
str(ab <- lapply(dimnames(qbBaha2017), as.numeric))
stopifnot(ab$a == c((1:6)/2, 5, 10, 25),
          ab$b == c(1:15, 10*c(2:5, 10, 25, 50)))
matplot(ab$b, t(qbBaha2017)[,9:1], type="l", log = "x", xlab = "b",
        ylab = "qbeta(.95, a,b)",
        main = "Guaranteed accuracy 95% percentiles of Beta distribution")
legend("right", paste("a = ", format(ab$a)),
       lty=1:5, col=1:6, bty="n")

## Relative error of R's qbeta() -- given that the table only shows 6
## digits, there is *no* relevant error: R's qbeta() is accurate enough:
x.ab <- do.call(expand.grid, ab)
matplot(ab$b, 1 - t(qbeta(0.95, x.ab$a, x.ab$b) / qbBaha2017),
        main = "rel.error of R's qbeta() -- w/ 6 digits, it is negligible",
        ylab = "1 - qbeta()/'true'",
        type = "l", log="x", xlab="b")
abline(h=0, col=adjustcolor("gray", 1/2))

```



stirlerrM

*Stirling Formula Approximation Error***Description**

Compute the `log()` of the error of Stirling's formula for  $n!$ . Used in certain accurate approximations of (negative) binomial and Poisson probabilities.

`stirlerrM()` currently simply uses the direct mathematical formula, based on `lgamma()`, adapted for use with `mpfr`-numbers.

**Usage**

```
stirlerrM(n, minPrec = 128L)
stirlerrSer(n, k)
```

**Arguments**

<code>n</code>	numeric or “numeric-alike” vector, typically “large” positive integer or half integer valued, here typically an “ <code>mpfr</code> ”-number vector.
<code>k</code>	integer <i>scalar</i> , now in 1:22.
<code>minPrec</code>	minimal precision (in bits) to be used when coercing number-alikes, say, biginteger ( <code>bigz</code> ) to “ <code>mpfr</code> ”.

**Details**

Stirling's approximation to  $n!$  has been

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n},$$

where by definition the error is the difference of the left and right hand side of this formula, in log-scale,

$$\delta(n) = \log \Gamma(n + 1) - n \log(n) + n - \log(2\pi n)/2.$$

See the vignette `log1pmx`, `bd0`, `stirlerr`, ... from package **DPQ**, where the series expansion of  $\delta(n)$  is used with 11 terms, starting with

$$\delta(n) = \frac{1}{12n} - \frac{1}{360n^3} + \frac{1}{1260n^5} \pm O(n^{-7}).$$

**Value**

a numeric or other “numeric-alike” class vector, e.g., `mpfr`, of the same length as `n`.

**Note**

In principle, the direct formula should be replaced by a few terms of the series in powers of  $1/n$  for large  $n$ , but we assume using high enough precision for  $n$  should be sufficient and “easier”.

**Author(s)**

Martin Maechler

**References**

Catherine Loader, see [dbinom](#);

Martin Maechler (2021) `log1pmx()`, `bd0()`, `stirlerr()` – Computing Poisson, Binomial, Gamma Probabilities in R. <https://CRAN.R-project.org/package=DPQ/vignettes/log1pmx-etc.pdf>

**See Also**

[dbinom](#); rational exact `dbinomQ()` in package **gmp**. `stirlerr()` in package **DPQ** which is a pure R version R's `mathlib-internal` C function.

**Examples**

```
### ----- Regular R double precision -----
n <- n. <- c(1:10, 15, 20, 30, 50*(1:6), 100*(4:9), 10^(3:12))
(stE <- stirlerrM(n)) # direct formula is *not* good when n is large:
require(graphics)
plot(stirlerrM(n) ~ n, log = "x", type = "b", xaxt="n") # --> *negative for large n!
sfsmisc::eaxis(1, sub10=3) ; abline(h = 0, lty=3, col=adjustcolor(1, 1/2))
oMax <- 22 # was 8 originally
str(stirSer <- sapply(setNames(1:oMax, paste0("k=", 1:oMax)),
                    function(k) stirlerrSer(n, k)))
cols <- 1:(oMax+1)
matlines(n, stirSer, col = cols, lty=1)
leg1 <- c("stirlerrM(n): [dbl prec] direct f()",
         paste0("stirlerrSer(n, k=", 1:oMax, ")"))
legend("top", leg1, pch = c(1,rep(NA,oMax)), col=cols, lty=1, bty="n")
## for larger n, current values are even *negative* ==> dbl prec *not* sufficient

## y in log-scale [same conclusion]
plot(stirlerrM(n) ~ n, log = "xy", type = "b", ylim = c(1e-13, 0.08))
matlines(n, stirSer, col = cols, lty=1)
legend("bottomleft", leg1, pch=c(1,rep(NA,oMax)), col=1:(oMax+1), lty=1, bty="n")

## the numbers:
options(digits=4, width=111)
stEmat. <- cbind(sM = setNames(stirlerrM(n),n), stirSer)
stEmat.
# note *bad* values for (n=1, k >= 8) !

## for printing n=<nice>: 8
N <- Rmpfr::asNumeric
dfm <- function(n, mm) data.frame(n=formatC(N(n)), N(mm), check.names=FALSE)
## relative differences:
dfm(n, stEmat.[,-1])/stEmat.[,1] - 1)
# => stirlerrM() {with dbl prec} deteriorates after ~ n = 200--500
```

```

### ----- MPFR High Accuracy -----

stopifnot(require(gmp),
           require(Rmpfr))
n <- as.bigz(n.)
## now repeat everything .. from above ... FIXME shows bugs !
## fully accurate using big rational arithmetic
class(stEserQ <- sapply(setNames(1:oMax, paste0("k=", 1:oMax)),
                       function(k) stirlerrSer(n=n, k=k))) # list ..
stopifnot(sapply(stEserQ, class) == "bigq") # of exact big rationals
str(stEsQM <- lapply(stEserQ, as, Class="mpfr"))# list of oMax; each prec. 128..702
stEsQM. <- lapply(stEserQ, .bigq2mpfr, precB = 512) # constant higher precision
stEsQMm <- sapply(stEserQ, asNumeric) # a matrix -- "exact" values of Stirling series

stEM <- stirlerrM(mpfr(n, 128)) # now ok (loss of precision, but still ~ 10 digits correct)
stEM4k <- stirlerrM(mpfr(n, 4096))# assume practically "perfect"/ "true" stirlerr() values
## ==> what's the accuracy of the 128-bit 'stEM'?
N <- asNumeric # short
dfm <- function(n, mm) data.frame(n=formatC(N(n)), N(mm), check.names=FALSE)
dfm(n, stEM/stEM4k - 1)
## .....
## 29 1e+06 4.470e-25
## 30 1e+07 -7.405e-23
## 31 1e+08 -4.661e-21
## 32 1e+09 -7.693e-20
## 33 1e+10 3.452e-17 (still ok)
## 34 1e+11 -3.472e-15 << now start losing --> 128 bits *not* sufficient!
## 35 1e+12 -3.138e-13 <<<<
## same conclusion via number of correct (decimal) digits:
dfm(n, log10(abs(stEM/stEM4k - 1)))

plot(N(-log10(abs(stEM/stEM4k - 1))) ~ N(n), type="o", log="x",
      xlab = quote(n), main = "{correct digits} of 128-bit stirlerrM(n)")
ubits <- c(128, 52) # above 128-bit and double precision
abline(h = ubits* log10(2), lty=2)
text(1, ubits* log10(2), paste0(ubits, "-bit"), adj=c(0,0))

stopifnot(identical(stirlerrM(n), stEM)) # for bigz & bigq, we default to precBits = 128
all.equal(roundMpfr(stEM4k, 64),
           stirlerrSer(n., oMax)) # 0.00212 .. because of 1st few n. ==> drop these
all.equal(roundMpfr(stEM4k, 64)[n. >= 3], stirlerrSer(n.[n. >= 3], oMax)) # 6.238e-8

plot(asNumeric(abs(stirlerrSer(n., oMax) - stEM4k)) ~ n.,
      log="xy", type="b", main="absolute error of stirlerrSer(n, oMax) & (n, 5)")
abline(h = 2^-52, lty=2); text(1, 2^-52, "52-bits", adj=c(1,-1)/oMax)
lines(asNumeric(abs(stirlerrSer(n., 5) - stEM4k)) ~ n., col=2)

plot(asNumeric(stirlerrM(n)) ~ n., log = "x", type = "b")
for(k in 1:oMax) lines(n, stirlerrSer(n, k), col = k+1)
legend("top", c("stirlerrM(n)", paste0("stirlerrSer(n, k=", 1:oMax, ")")),

```

```

pch=c(1,rep(NA,oMax)), col=1:(oMax+1), lty=1, bty="n")

## y in log-scale
plot(asNumeric(stirlerrM(n)) ~ n., log = "xy", type = "b", ylim = c(1e-13, 0.08))
for(k in 1:oMax) lines(n, stirlerrSer(n, k), col = k+1)
legend("topright", c("stirlerrM(n)", paste0("stirlerrSer(n, k=", 1:oMax, ")")),
      pch=c(1,rep(NA,oMax)), col=1:(oMax+1), lty=1, bty="n")
## all "looks" perfect (so we could skip this)

## The numbers ... reused
## stopifnot(sapply(stEserQ, class) == "bigq") # of exact big rationals
## str(stEsQM <- lapply(stEserQ, as, Class="mpfr"))# list of oMax; each prec. 128..702
##   stEsQM. <- lapply(stEserQ, .bigq2mpfr, precB = 512) # constant higher precision
##   stEsQMm <- sapply(stEserQ, asNumeric) # a matrix -- "exact" values of Stirling series

## stEM <- stirlerrM(mpfr(n, 128)) # now ok (loss of precision, but still ~ 10 digits correct)
## stEM4k <- stirlerrM(mpfr(n, 4096))# assume "perfect"
stEmat <- cbind(sM = stEM4k, stEsQMm)
signif(asNumeric(stEmat), 6) # prints nicely -- large n = 10^e: see ~ = 1/(12 n) = 0.8333 / n
## print *relative errors* nicely :
## simple double precision version of direct formula (cancellation for n >> 1 !):
stE <- stirlerrM(n.) # --> bad for small n; catastrophically bad for n >= 10^7
## relative *errors*
dfm(n , cbind(stEsQMm, dbl=stE)/stEM4k - 1)
## only "perfect" Series (showing true mathematical approx. error; not *numerical*
relE <- N(stEsQMm / stEM4k - 1)
dfm(n, relE)
matplot(n, relE, type = "b", log="x", ylim = c(-1,1) * 1e-12)
## |rel.Err| in [log log]
matplot(n, abs(N(relE)), type = "b", log="xy")
matplot(n, pmax(abs(N(relE)), 1e-19), type = "b", log="xy", ylim = c(1e-17, 1e-12))
matplot(n, pmax(abs(N(relE)), 1e-19), type = "b", log="xy", ylim = c(4e-17, 1e-15))
abline(h = 2^-(53:52), lty=3)

```

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