Solving the $N$-Queens Problem with Local Search

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This vignette provides example code for a combinatorial problem: the $N$-Queens Problem.

1 The problem

The goal is to place $N$ queens on a chess-board of size $N \times N$ in such a way that no queen is attacked. A queen may move vertically, horizontally and on a diagonal. So whenever there is more than one queen on any row, column or diagonal, the position is invalid. To solve the problem with a Local Search (LS), we need three components:

1. a way to represent a solution (i.e. a position on the chessboard);
2. a way to evaluate such a solution;
3. and, since we use a LS, a method to modify a solution.

We start by attaching the package and fixing a seed.

> library("NMOF")
> set.seed(134577)

2 Representing a solution

Since on any row there cannot be more than one queen, we may store a position as a vector of columns on which the queens are placed. (In chess, rows would be called ranks and columns would be files, but we prefer matrix terminology.) Thus, a candidate solution $p$ (p for position) could look as follows:

> N <- 8 ## board size
> p <- sample.int(N) ## a random solution
> data.frame(row = 1:N, column = p)

<table>
<thead>
<tr>
<th>row</th>
<th>column</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>8</td>
</tr>
<tr>
<td>2</td>
<td>7</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
</tr>
<tr>
<td>7</td>
<td>5</td>
</tr>
<tr>
<td>8</td>
<td>4</td>
</tr>
</tbody>
</table>

Or (a very bad solution):

> p <- rep(1, N)
> data.frame(row = 1:N, column = p)

<table>
<thead>
<tr>
<th>row</th>
<th>column</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
</tr>
<tr>
<td>8</td>
<td>1</td>
</tr>
</tbody>
</table>
We will also want to visualise a position, for which we write the function `print_board`.

```r
> print_board <- function(p, q.char = "Q", sep = " ") {
    n <- length(p)
    row <- rep("-", n)
    for (i in seq_len(n)) {
        row_i <- row
        row_i[p[i]] <- q.char
        cat(paste(row_i, collapse = sep))
    }
    cat("\n")
}
> print_board(p)
Q - - - - - - -
Q - - - - - - -
Q - - - - - - -
Q - - - - - - -
Q - - - - - - -
Q - - - - - - -
Q - - - - - - -
Q - - - - - - -
```

3 Evaluating a solution

We need to compute on what row, column, diagonal (top left to bottom right) or reverse diagonal (top right to bottom left) a queen stands. Rows and columns are simple; we label the diagonals as follows.

```r
> mat <- array(NA, dim = c(N,N)) ## diagonals
> for (r in 1:N)
    for (c in 1:N)
        mat[r,c] <- c - r
> mat
[1,]  0  1  2  3  4  5  6  7
[2,] -1  0  1  2  3  4  5  6
[3,] -2 -1  0  1  2  3  4  5
[4,] -3 -2 -1  0  1  2  3  4
[5,] -4 -3 -2 -1  0  1  2  3
[6,] -5 -4 -3 -2 -1  0  1  2
[7,] -6 -5 -4 -3 -2 -1  0  1
[8,] -7 -6 -5 -4 -3 -2 -1  0
```

```r
> mat <- array(NA, dim = c(N,N)) ## reverse diagonals
> for (r in 1:N)
    for (c in 1:N)
        mat[r,c] <- c + r - (N + 1)
> mat
[1,] -7 -6 -5 -4 -3 -2 -1  0
[2,] -6 -5 -4 -3 -2 -1  0  1
[3,] -5 -4 -3 -2 -1  0  1  2
[4,] -4 -3 -2 -1  0  1  2  3
[5,] -3 -2 -1  0  1  2  3  4
[6,] -2 -1  0  1  2  3  4  5
[7,] -1  0  1  2  3  4  5  6
[8,]  0  1  2  3  4  5  6  7
```
Note that for reverse diagonals, the $N + 1$ would not be necessary; it serves only to shift the diagonal labels so that the main diagonal is zero.

Thus for a given solution $p$, we know the row, column, diagonal and reverse diagonal for each queen. We define the quality of a solution by the number of attacks that happen: for a valid solution, that number should be zero.

```r
> n_attacks <- function(p) {
    ## more than one Q on a column?
    sum(duplicated(p)) +

    ## more than one Q on a diagonal?
    sum(duplicated(p - seq_along(p))) +

    ## more than one Q on a reverse diagonal?
    sum(duplicated(p + seq_along(p)))
}
> n_attacks(p)

[1] 7
```

### 4 Changing a solution

A given position may be modified by picking one row randomly and then moving the queen there to the left or right. We allow for moves up to $\text{step}$ squares, which we set to 3 in the example.

```r
> neighbour <- function(p) {
    step <- 3
    i <- sample.int(N, 1)
    p[i] <- p[i] + sample(c(1:step, -(1:step)), 1)

    if (p[i] > N)
        p[i] <- 1
    else if (p[i] < 1)
        p[i] <- N
    p
}

> print_board(p)

Q - - - - - - -
Q - - - - - - -
Q - - - - - - -
Q - - - - - - -
Q - - - - - - -
Q - - - - - - -
Q - - - - - - -
Q - - - - - - -

> print_board(p <- neighbour(p))

Q - - - - - - -
Q - - - - - - -
- - - Q - - - -
Q - - - - - - -
Q - - - - - - -
Q - - - - - - -
Q - - - - - - -
Q - - - - - - -
```
5 Solving the model

We use three different LS methods: a ‘classical’ Stochastic Local Search (LSopt), Threshold Accepting (TAopt) and Simulated Annealing (SAopt).

```r
> p0 <- rep(1, N) ## or a random initial solution: p0 <- sample.int(N)
> print_board(p0)
Q - - - - - - -
Q - - - - - - -
- - - Q - - - -
Q - - - - - - -
Q - - - - - - -
- - - - - - - Q
Q - - - - - - -
Q - - - - - - -
Q - - - - - - -

> sol <- LSopt(n_attacks, list(x0 = p0,
                          neighbour = neighbour,
                          printBar = FALSE,
                          nS = 10000))

Local Search.
Initial solution: 7
Finished.
Best solution overall: 0

> print_board(sol$xbest)
- Q - - - - - -
- - - - - - Q -
- - Q - - - - -
- - - - - Q - -
- - - - - - - Q
- - - - Q - - -
Q - - - - - - -
- - Q - - - - -

> sol <- TAopt(n_attacks, list(x0 = p0,
                               neighbour = neighbour,
                               printBar = FALSE,
                               nS = 1000))

Threshold Accepting
Computing thresholds ... OK
Estimated remaining running time: 0.185 secs
Running Threshold Accepting ...
Initial solution: 7
Finished.
Best solution overall: 0

> print_board(sol$xbest)
- - - - Q - - -
- - - - - - Q -
- Q - - - - - -
- - - Q - - - -
- - - - - Q - -
- - - - - - - Q
Q - - - - - - -
- - Q - - - - -
- - - - - Q - -

> sol <- SAopt(n_attacks, list(x0 = p0,
neighbour = neighbour,
printBar = FALSE,
nS = 1000))

Simulated Annealing.
Calibrating acceptance criterion ... OK
Estimated remaining running time: 0.19 secs.
Running Simulated Annealing ... 
Initial solution: 7 
Finished.
Best solution overall: 0

> print_board(sol$xbest)
- - - - Q - - -
- - - - - - Q -
- Q - - - - - -
- - - Q - - - -
- - - - - Q - -
- - - - - - - Q
Q - - - - - - -
- - Q - - - - -
- - - - - Q - -

References