Solving the $N$-Queens Problem with Local Search

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This vignette provides example code for a combinatorial problem: the $N$-Queens Problem.

\section{The problem}

The goal is to place $N$ queens on a chess-board of size $N \times N$ in such a way that no queen is attacked. A queen may move vertically, horizontally and on a diagonal. So whenever there is more than one queen on any row, column or diagonal, the position is invalid. To solve the problem with an Local Search (LS), we need three components:

1. a way to represent a solution (i.e. a position on the chessboard);

2. a way to evaluate such a solution;

3. and, since we use a LS, a method to modify a solution.

We start by attaching the package and fixing a seed.

\begin{verbatim}\mtext{\> library("NMOF") \> set.seed(134577)}\end{verbatim}

\section{Representing a solution}

Since on any row there cannot be more than one queen, we may store a position as a vector of columns on which the queens are placed. (In chess, rows would be called ranks and columns would be files, but we prefer matrix terminology.) Thus, a candidate solution $p$ ($p$ for position) could look as follows:

\begin{verbatim}\mtext{\> N <- 8 \> p <- sample.int(N) \> data.frame(row = 1:N, column = p)}\end{verbatim}

\begin{verbatim}\begin{tabular}{ll}
row & column \\
1 & 1 \\
2 & 2 \\
3 & 3 \\
4 & 4 \\
5 & 5 \\
6 & 6 \\
7 & 7 \\
8 & 8 \\
\end{tabular}\end{verbatim}

Or (a very bad solution):

\begin{verbatim}\mtext{\> p <- rep(1, N) \> data.frame(row = 1:N, column = p)}\end{verbatim}

\begin{verbatim}\begin{tabular}{ll}
row & column \\
1 & 1 \\
2 & 2 \\
3 & 3 \\
4 & 4 \\
5 & 5 \\
6 & 6 \\
7 & 7 \\
8 & 8 \\
\end{tabular}\end{verbatim}
We may also want to visualise a position, for which may use the function `print_board`.

```r
> print_board <- function(p, q.char = "Q", sep = " ") {
  n <- length(p)
  row <- rep("-", n)
  for (i in seq_len(n)) {
    row_i <- row
    row_i[p[i]] <- q.char
    cat(paste(row_i, collapse = sep))
  }
  cat("\n")
}

> print_board(p)
Q - - - - - - -
Q - - - - - - -
Q - - - - - - -
Q - - - - - - -
Q - - - - - - -
Q - - - - - - -
Q - - - - - - -
Q - - - - - - -
```

3 Evaluating a solution

We need to compute on what row, column, diagonal (top left to bottom right) or reverse diagonal (top right to bottom left) a queen stands. Rows and columns are simple; we label the diagonals as follows.

```r
> mat <- array(NA, dim = c(N,N)) ## diagonals
> for (r in 1:N) # diagonals
  for (c in 1:N)
    mat[r,c] <- c - r

> mat
[1,]  0  1  2  3  4  5  6  7
[2,] -1  0  1  2  3  4  5  6
[3,] -2 -1  0  1  2  3  4  5
[4,] -3 -2 -1  0  1  2  3  4
[5,] -4 -3 -2 -1  0  1  2  3
[6,] -5 -4 -3 -2 -1  0  1  2
[7,] -6 -5 -4 -3 -2 -1  0  1
[8,] -7 -6 -5 -4 -3 -2 -1  0
```

```r
> mat <- array(NA, dim = c(N,N)) ## reverse diagonals
> for (r in 1:N) # reverse diagonals
  for (c in 1:N)
    mat[r,c] <- c + r - (N + 1)

> mat
[1,] -7 -6 -5 -4 -3 -2 -1  0
[2,] -6 -5 -4 -3 -2 -1  0  1
[3,] -5 -4 -3 -2 -1  0  1  2
[4,] -4 -3 -2 -1  0  1  2  3
[5,] -3 -2 -1  0  1  2  3  4
[6,] -2 -1  0  1  2  3  4  5
[7,] -1  0  1  2  3  4  5  6
[8,]  0  1  2  3  4  5  6  7
```
Note that for reverse diagonals, the $N + 1$ would not be necessary; it serves only to shift the diagonal labels so that the main diagonal is zero.

Thus for a given solution $p$, we know the row, column, diagonal and reverse diagonal for each queen. We define the quality of a solution by the number of attacks that happen: for a valid solution, that number should be zero.

```r
> n_attacks <- function(p) {
  ## more than one Q on a column?
  sum(duplicated(p)) +

  ## more than one Q on a diagonal?
  sum(duplicated(p - seq_along(p))) +

  ## more than one Q on a reverse diagonal?
  sum(duplicated(p + seq_along(p)))
}
```

```r
> n_attacks(p)
[1] 7
```

4 Changing a solution

A given position may be modified by picking one row randomly and then moving the queen there to the left or right. We allow for moves up to $\text{step}$ squares, which we set to 3 in the example.

```r
> neighbour <- function(p) {
  step <- 3
  i <- sample.int(N, 1)
  p[i] <- p[i] + sample(c(1:step, -(1:step)), 1)

  if (p[i] > N)
    p[i] <- 1
  else if (p[i] < 1)
    p[i] <- N
  p
}
```

```r
> print_board(p)
  Q - - - - - - -
  Q - - - - - - -
  Q - - - - - - -
  Q - - - - - - -
  Q - - - - - - -
  Q - - - - - - -
  Q - - - - - - -
  Q - - - - - - -

> print_board(p <- neighbour(p))
  Q - - - - - - -
  Q - - - - - - -
  - - - Q - - - -
  Q - - - - - - -
  Q - - - - - - -
  Q - - - - - - -
  Q - - - - - - -
  Q - - - - - - -
```
\section{Solving the model}

We use three different LS methods: a ‘classical’ Stochastic Local Search (LSopt), Threshold Accepting (TAopt) and Simulated Annealing (SAopt).

\begin{verbatim}
> p0 <- rep(1, N) ## or a random initial solution: p0 <- sample.int(N)
> print_board(p0)

Q - - - - - - -
Q - - - - - - -
- - - Q - - -
Q - - - - - - -
Q - - - - - - -
- - - - - - - Q
Q - - - - - - -
Q - - - - - - -
Q - - - - - - -

> sol <- LSopt(n_attacks, list(x0 = p0,
neighbour = neighbour,
printBar = FALSE,
nS = 10000))
Local Search.
Initial solution: 7
Finished.
Best solution overall: 0

> print_board(sol$xbest)

- Q - - - - -
- - - - - - Q-
- - Q - - - -
- - - - - - Q-
- - - - - - Q-
Q - - - - - - -
- - Q - - - -

> sol <- TAopt(n_attacks, list(x0 = p0,
neighbour =ighbour,
printBar = FALSE,
nS = 1000))
Threshold Accepting
Computing thresholds ... OK
Estimated remaining running time: 0.23 secs
Running Threshold Accepting ...
Initial solution: 7
\end{verbatim}
Finished.
Best solution overall: 0

> print_board(sol$xbest)

- - - - Q - - -
- - - - - - Q -
- Q - - - - - -
- - - Q - - - -
- - - - - - - Q
 Q - - - - - - -
- - Q - - - - -
- - - - - - Q -
- - - - - - - Q

> sol <- SAopt(n_attacks, list(x0 = p0, 
neighbour = neighbour, 
printBar = FALSE, 
nS = 1000))

Simulated Annealing.
Calibrating acceptance criterion ... OK
Estimated remaining running time: 0.225 secs.

Running Simulated Annealing ...
Initial solution: 7
Finished.
Best solution overall: 0

> print_board(sol$xbest)

- - - - Q - - -
- - - - - - Q -
- Q - - - - - -
- - - Q - - - -
- - - - - - - Q
 Q - - - - - - -
- - Q - - - - -
- - - - - - Q -
- - - - - - - Q

References