

Package ‘energy’

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Title E-Statistics: Multivariate Inference via the Energy of Data

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Description E-statistics (energy) tests and statistics for multivariate and univariate inference, including distance correlation, one-sample, two-sample, and multi-sample tests for comparing multivariate distributions, are implemented. Measuring and testing multivariate independence based on distance correlation, partial distance correlation, multivariate goodness-of-fit tests, clustering based on energy distance, testing for multivariate normality, distance components (disco) for non-parametric analysis of structured data, and other energy statistics/methods are implemented.

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energy-package

E-statistics: Multivariate Inference via the Energy of Data

Description

Description: E-statistics (energy) tests and statistics for multivariate and univariate inference, including distance correlation, one-sample, two-sample, and multi-sample tests for comparing multivariate distributions, are implemented. Measuring and testing multivariate independence based on distance correlation, partial distance correlation, multivariate goodness-of-fit tests, clustering based on energy distance, testing for multivariate normality, distance components (disco) for non-parametric analysis of structured data, and other energy statistics/methods are implemented.

Author(s)

Maria L. Rizzo and Gabor J. Szekely

References

- G. J. Szekely and M. L. Rizzo (2013). Energy statistics: A class of statistics based on distances, *Journal of Statistical Planning and Inference*, <http://dx.doi.org/10.1016/j.jspi.2013.03.018>
- M. L. Rizzo and G. J. Szekely (2016). Energy Distance, *WIRES Computational Statistics*, Wiley, Volume 8 Issue 1, 27-38. Available online Dec., 2015, <http://dx.doi.org/10.1002/wics.1375>.
- G. J. Szekely and M. L. Rizzo (2017). The Energy of Data. *The Annual Review of Statistics and Its Application* 4:447-79. [10.1146/annurev-statistics-060116-054026](https://doi.org/10.1146/annurev-statistics-060116-054026)

 centering distance matrices

Double centering and U-centering

Description

Stand-alone double centering and U-centering functions that are applied in unbiased distance covariance, bias corrected distance correlation, and partial distance correlation.

Usage

```
Dcenter(x)
Ucenter(x)
U_center(Dx)
D_center(Dx)
```

Arguments

x	dist object or data matrix
Dx	distance or dissimilarity matrix

Details

In `Dcenter` and `Ucenter`, `x` must be a `dist` object or a data matrix. Both functions return a doubly centered distance matrix.

Note that `pdcor`, etc. functions include the centering operations (in `C`), so that these stand alone versions of centering functions are not needed except in case one wants to compute just a double-centered or U-centered matrix.

`U_center` is the Rcpp export of the `cpp` function. `D_center` is the Rcpp export of the `cpp` function.

Value

All functions return a square symmetric matrix.

`Dcenter` returns a matrix

$$A_{ij} = a_{ij} - \bar{a}_{i.} - \bar{a}_{.j} + \bar{a}_{..}$$

as in classical multidimensional scaling. `Ucenter` returns a matrix

$$\tilde{A}_{ij} = a_{ij} - \frac{a_{i.}}{n-2} - \frac{a_{.j}}{n-2} + \frac{a_{..}}{(n-1)(n-2)}, \quad i \neq j,$$

with zero diagonal, and this is the double centering applied in `pdcov` and `pdcor` as well as the unbiased `dCov` and bias corrected `dCor` statistics.

Note

The c++ versions `D_center` and `U_center` should typically be faster. R versions are retained for historical reasons.

Author(s)

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References

Szekely, G.J. and Rizzo, M.L. (2014), Partial Distance Correlation with Methods for Dissimilarities, *Annals of Statistics*, Vol. 42, No. 6, pp. 2382-2412. DOI [dx.doi.org/10.1214/14-AOS1255](https://doi.org/10.1214/14-AOS1255)
<http://projecteuclid.org/euclid.aos/1413810731>

Examples

```
x <- iris[1:10, 1:4]
dx <- dist(x)
Dx <- as.matrix(dx)
M <- U_center(Dx)

all.equal(M, U_center(M))    #idempotence
all.equal(M, D_center(M))    #invariance
```

dcor.ttest

Distance Correlation t-Test

Description

Distance correlation t-test of multivariate independence.

Usage

```
dcor.ttest(x, y, distance=FALSE)
dcor.t(x, y, distance=FALSE)
```

Arguments

x	data or distances of first sample
y	data or distances of second sample
distance	logical: TRUE if x and y are distances

Details

dcor.ttest performs a nonparametric t-test of multivariate independence in high dimension (dimension is close to or larger than sample size). The distribution of the test statistic is approximately Student t with $n(n - 3)/2 - 1$ degrees of freedom and for $n \geq 10$ the statistic is approximately distributed as standard normal.

The sample sizes (number of rows) of the two samples must agree, and samples must not contain missing values. Arguments x, y can optionally be dist objects or distance matrices (in this case set distance=TRUE).

The t statistic is a transformation of a bias corrected version of distance correlation (see SR 2013 for details).

Large values (upper tail) of the t statistic are significant.

Value

dcor.t returns the t statistic, and dcor.ttest returns a list with class htest containing

method	description of test
statistic	observed value of the test statistic
parameter	degrees of freedom
estimate	(bias corrected) dCor(x,y)
p.value	p-value of the t-test
data.name	description of data

Author(s)

Maria L. Rizzo <mrizzo @ bgsu.edu> and Gabor J. Szekely

References

Szekely, G.J. and Rizzo, M.L. (2013). The distance correlation t-test of independence in high dimension. *Journal of Multivariate Analysis*, Volume 117, pp. 193-213.

<http://dx.doi.org/10.1016/j.jmva.2013.02.012>

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<http://dx.doi.org/10.1214/009053607000000505>

Szekely, G.J. and Rizzo, M.L. (2009), Brownian Distance Covariance, *Annals of Applied Statistics*, Vol. 3, No. 4, 1236-1265.

<http://dx.doi.org/10.1214/09-AOAS312>

See Also

[bcdcor](#) [dcov.test](#) [dcor](#) [DCOR](#)

Examples

```
x <- matrix(rnorm(100), 10, 10)
y <- matrix(runif(100), 10, 10)
dx <- dist(x)
dy <- dist(y)
dcor.t(x, y)
dcor.ttest(x, y)
```

 dcov.test

Distance Covariance Test and Distance Correlation test

Description

Distance covariance test and distance correlation test of multivariate independence. Distance covariance and distance correlation are multivariate measures of dependence.

Usage

```
dcov.test(x, y, index = 1.0, R = NULL)
dcor.test(x, y, index = 1.0, R)
```

Arguments

x	data or distances of first sample
y	data or distances of second sample
R	number of replicates
index	exponent on Euclidean distance, in (0,2]

Details

dcov.test and dcor.test are nonparametric tests of multivariate independence. The test decision is obtained via permutation bootstrap, with R replicates.

The sample sizes (number of rows) of the two samples must agree, and samples must not contain missing values. Arguments x, y can optionally be `dist` objects; otherwise these arguments are treated as data.

The dcov test statistic is $n\mathcal{V}_n^2$ where $\mathcal{V}_n(x, y) = \text{dcov}(x, y)$, which is based on interpoint Euclidean distances $\|x_i - x_j\|$. The index is an optional exponent on Euclidean distance.

Similarly, the dcor test statistic is based on the normalized coefficient, the distance correlation. (See the manual page for dcor.)

Distance correlation is a new measure of dependence between random vectors introduced by Szekely, Rizzo, and Bakirov (2007). For all distributions with finite first moments, distance correlation \mathcal{R} generalizes the idea of correlation in two fundamental ways:

- (1) $\mathcal{R}(X, Y)$ is defined for X and Y in arbitrary dimension.
- (2) $\mathcal{R}(X, Y) = 0$ characterizes independence of X and Y .

Characterization (2) also holds for powers of Euclidean distance $\|x_i - x_j\|^s$, where $0 < s < 2$, but (2) does not hold when $s = 2$.

Distance correlation satisfies $0 \leq \mathcal{R} \leq 1$, and $\mathcal{R} = 0$ only if X and Y are independent. Distance covariance \mathcal{V} provides a new approach to the problem of testing the joint independence of random vectors. The formal definitions of the population coefficients \mathcal{V} and \mathcal{R} are given in (SRB 2007). The definitions of the empirical coefficients are given in the energy `dcov` topic.

For all values of the index in $(0,2)$, under independence the asymptotic distribution of $n\mathcal{V}_n^2$ is a quadratic form of centered Gaussian random variables, with coefficients that depend on the distributions of X and Y . For the general problem of testing independence when the distributions of X and Y are unknown, the test based on $n\mathcal{V}_n^2$ can be implemented as a permutation test. See (SRB 2007) for theoretical properties of the test, including statistical consistency.

Value

`dcov.test` or `dcor.test` returns a list with class `htest` containing

<code>method</code>	description of test
<code>statistic</code>	observed value of the test statistic
<code>estimate</code>	<code>dCov(x,y)</code> or <code>dCor(x,y)</code>
<code>estimates</code>	a vector: [<code>dCov(x,y)</code> , <code>dCor(x,y)</code> , <code>dVar(x)</code> , <code>dVar(y)</code>]
<code>replicates</code>	replicates of the test statistic
<code>p.value</code>	approximate p-value of the test
<code>n</code>	sample size
<code>data.name</code>	description of data

Note

For the `dcov` test of independence, the distance covariance test statistic is the V-statistic $n \text{dCov}^2 = n\mathcal{V}_n^2$ (not `dCov`).

Author(s)

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References

- Szekely, G.J., Rizzo, M.L., and Bakirov, N.K. (2007), Measuring and Testing Dependence by Correlation of Distances, *Annals of Statistics*, Vol. 35 No. 6, pp. 2769-2794.
<http://dx.doi.org/10.1214/009053607000000505>
- Szekely, G.J. and Rizzo, M.L. (2009), Brownian Distance Covariance, *Annals of Applied Statistics*, Vol. 3, No. 4, 1236-1265.
<http://dx.doi.org/10.1214/09-AOAS312>
- Szekely, G.J. and Rizzo, M.L. (2009), Rejoinder: Brownian Distance Covariance, *Annals of Applied Statistics*, Vol. 3, No. 4, 1303-1308.

See Also

[dcov](#) [dcor](#) [DCOR](#) [dcor.ttest](#)

Examples

```

x <- iris[1:50, 1:4]
y <- iris[51:100, 1:4]
set.seed(1)
dcor.test(dist(x), dist(y), R=199)
set.seed(1)
dcov.test(x, y, R=199)

## Not run:
set.seed(1)
dcov.test(dist(x), dist(y), R=199) #same thing
set.seed(1)
dcov.test(x, y, index=.5, R=199)
set.seed(1)
dcov.test(dist(x), dist(y), index=.5, R=199) #same thing

## End(Not run)

## Example with dvar=0 (so dcov=0 and pval=1)
x <- rep.int(1, 10)
y <- 1:10
dcov.test(x, y, R=199)

```

dcovU_stats

Unbiased distance covariance statistics

Description

This function computes unbiased estimators of squared distance covariance, distance variance, and a bias-corrected estimator of (squared) distance correlation.

Usage

```
dcovU_stats(Dx, Dy)
```

Arguments

Dx	distance matrix of first sample
Dy	distance matrix of second sample

Details

The unbiased (squared) dcov is inner product definition of dCov, in the Hilbert space of U-centered distance matrices.

The sample sizes (number of rows) of the two samples must agree, and samples must not contain missing values. The arguments must be square symmetric matrices.

Value

dcovU_stats returns a vector of the components of bias-corrected dcor: [dCovU, bcdcor, dVarXU, dVarYU].

Note

Unbiased distance covariance (SR2014) corresponds to the biased (original) dCov². Since dcovU is an unbiased statistic, it is signed and we do not take the square root. For the original distance covariance test of independence (SRB2007, SR2009), the distance covariance test statistic is the V-statistic $n \text{dCov}^2 = n\mathcal{V}_n^2$ (not dCov). Similarly, bcdcor is bias-corrected, so we do not take the square root as with dCor.

Author(s)

Maria L. Rizzo <mrizzo @ bgsu.edu> and Gabor J. Szekely

References

Szekely, G.J. and Rizzo, M.L. (2014), Partial Distance Correlation with Methods for Dissimilarities. *Annals of Statistics*, Vol. 42 No. 6, 2382-2412.

Szekely, G.J., Rizzo, M.L., and Bakirov, N.K. (2007), Measuring and Testing Dependence by Correlation of Distances, *Annals of Statistics*, Vol. 35 No. 6, pp. 2769-2794.
<http://dx.doi.org/10.1214/009053607000000505>

Szekely, G.J. and Rizzo, M.L. (2009), Brownian Distance Covariance, *Annals of Applied Statistics*, Vol. 3, No. 4, 1236-1265.
<http://dx.doi.org/10.1214/09-AOAS312>

Examples

```
x <- iris[1:50, 1:4]
y <- iris[51:100, 1:4]
Dx <- as.matrix(dist(x))
Dy <- as.matrix(dist(y))
dcovU_stats(Dx, Dy)
```

 disco

distance components (DISCO)

Description

E-statistics DIStance COmponents and tests, analogous to variance components and anova.

Usage

```
disco(x, factors, distance, index=1.0, R, method=c("disco","discoB","discoF"))
disco.between(x, factors, distance, index=1.0, R)
```

Arguments

x	data matrix or distance matrix or dist object
factors	matrix of factor labels or integers (not design matrix)
distance	logical, TRUE if x is distance matrix
index	exponent on Euclidean distance in (0,2]
R	number of replicates for a permutation test
method	test statistic

Details

disco calculates the distance components decomposition of total dispersion and if $R > 0$ tests for significance using the test statistic disco "F" ratio (default method="disco"), or using the between component statistic (method="discoB"), each implemented by permutation test.

If x is a dist object, argument distance is ignored. If x is a distance matrix, set distance=TRUE.

In the current release disco computes the decomposition for one-way models only.

Value

When method="discoF", disco returns a list similar to the return value from `anova.lm`, and the `print.disco` method is provided to format the output into a similar table. Details:

disco returns a class disco object, which is a list containing

call	call
method	method
statistic	vector of observed statistics
p.value	vector of p-values
k	number of factors
N	number of observations
between	between-sample distance components
within	one-way within-sample distance components
total	total dispersion
Df.trt	degrees of freedom for treatments
Df.e	degrees of freedom for error
index	index (exponent on distance)
factor.names	factor names
factor.levels	factor levels
sample.sizes	sample sizes
stats	matrix containing decomposition

When method="discoB", disco passes the arguments to `disco.between`, which returns a class `hstest` object.

`disco.between` returns a class `hstest` object, where the test statistic is the between-sample statistic (proportional to the numerator of the F ratio of the disco test).

Note

The current version does all calculations via matrix arithmetic and boot function. Support for more general additive models and a formula interface is under development.

disco methods have been added to the cluster distance summary function `edist`, and energy tests for equality of distribution (see `eqdist.etest`).

Author(s)

Maria L. Rizzo <`mrizzo @ bgsu.edu`> and Gabor J. Szekely

References

M. L. Rizzo and G. J. Szekely (2010). DISCO Analysis: A Nonparametric Extension of Analysis of Variance, *Annals of Applied Statistics*, Vol. 4, No. 2, 1034-1055.

<http://dx.doi.org/10.1214/09-AOAS245>

See Also

`edist` `eqdist.e` `eqdist.etest` `ksample.e`

Examples

```
## warpbreaks one-way decompositions
data(warpbreaks)
attach(warpbreaks)
disco(breaks, factors=wool, R=99)

## When index=2 for univariate data, we get ANOVA decomposition
disco(breaks, factors=tension, index=2.0, R=99)
aov(breaks ~ tension)

## Multivariate response
## Example on producing plastic film from Krzanowski (1998, p. 381)
tear <- c(6.5, 6.2, 5.8, 6.5, 6.5, 6.9, 7.2, 6.9, 6.1, 6.3,
          6.7, 6.6, 7.2, 7.1, 6.8, 7.1, 7.0, 7.2, 7.5, 7.6)
gloss <- c(9.5, 9.9, 9.6, 9.6, 9.2, 9.1, 10.0, 9.9, 9.5, 9.4,
          9.1, 9.3, 8.3, 8.4, 8.5, 9.2, 8.8, 9.7, 10.1, 9.2)
opacity <- c(4.4, 6.4, 3.0, 4.1, 0.8, 5.7, 2.0, 3.9, 1.9, 5.7,
            2.8, 4.1, 3.8, 1.6, 3.4, 8.4, 5.2, 6.9, 2.7, 1.9)
Y <- cbind(tear, gloss, opacity)
rate <- factor(gl(2,10), labels=c("Low", "High"))

## test for equal distributions by rate
disco(Y, factors=rate, R=99)
disco(Y, factors=rate, R=99, method="discoB")

## Just extract the decomposition table
disco(Y, factors=rate, R=0)$stats

## Compare eqdist.e methods for rate
## disco between stat is half of original when sample sizes equal
```

```

eqdist.e(Y, sizes=c(10, 10), method="original")
eqdist.e(Y, sizes=c(10, 10), method="discoB")

## The between-sample distance component
disco.between(Y, factors=rate, R=0)

```

distance correlation *Distance Correlation and Covariance Statistics*

Description

Computes distance covariance and distance correlation statistics, which are multivariate measures of dependence.

Usage

```

dcov(x, y, index = 1.0)
dcor(x, y, index = 1.0)
DCOR(x, y, index = 1.0)

```

Arguments

x	data or distances of first sample
y	data or distances of second sample
index	exponent on Euclidean distance, in (0,2]

Details

dcov and dcor or DCOR compute distance covariance and distance correlation statistics. DCOR is a self-contained R function returning a list of statistics. dcor execution is faster than DCOR (see examples).

The sample sizes (number of rows) of the two samples must agree, and samples must not contain missing values. Arguments x, y can optionally be `dist` objects; otherwise these arguments are treated as data.

Distance correlation is a new measure of dependence between random vectors introduced by Szekely, Rizzo, and Bakirov (2007). For all distributions with finite first moments, distance correlation \mathcal{R} generalizes the idea of correlation in two fundamental ways: (1) $\mathcal{R}(X, Y)$ is defined for X and Y in arbitrary dimension. (2) $\mathcal{R}(X, Y) = 0$ characterizes independence of X and Y .

Distance correlation satisfies $0 \leq \mathcal{R} \leq 1$, and $\mathcal{R} = 0$ only if X and Y are independent. Distance covariance \mathcal{V} provides a new approach to the problem of testing the joint independence of random vectors. The formal definitions of the population coefficients \mathcal{V} and \mathcal{R} are given in (SRB 2007). The definitions of the empirical coefficients are as follows.

The empirical distance covariance $\mathcal{V}_n(\mathbf{X}, \mathbf{Y})$ with index 1 is the nonnegative number defined by

$$\mathcal{V}_n^2(\mathbf{X}, \mathbf{Y}) = \frac{1}{n^2} \sum_{k,l=1}^n A_{kl} B_{kl}$$

where A_{kl} and B_{kl} are

$$A_{kl} = a_{kl} - \bar{a}_{k.} - \bar{a}_{.l} + \bar{a}_{..}$$

$$B_{kl} = b_{kl} - \bar{b}_{k.} - \bar{b}_{.l} + \bar{b}_{..}$$

Here

$$a_{kl} = \|X_k - X_l\|_p, \quad b_{kl} = \|Y_k - Y_l\|_q, \quad k, l = 1, \dots, n,$$

and the subscript \cdot denotes that the mean is computed for the index that it replaces. Similarly, $\mathcal{V}_n(\mathbf{X})$ is the nonnegative number defined by

$$\mathcal{V}_n^2(\mathbf{X}) = \mathcal{V}_n^2(\mathbf{X}, \mathbf{X}) = \frac{1}{n^2} \sum_{k,l=1}^n A_{kl}^2.$$

The empirical distance correlation $\mathcal{R}_n(\mathbf{X}, \mathbf{Y})$ is the square root of

$$\mathcal{R}_n^2(\mathbf{X}, \mathbf{Y}) = \frac{\mathcal{V}_n^2(\mathbf{X}, \mathbf{Y})}{\sqrt{\mathcal{V}_n^2(\mathbf{X})\mathcal{V}_n^2(\mathbf{Y})}}.$$

See [dcov.test](#) for a test of multivariate independence based on the distance covariance statistic.

Value

dcov returns the sample distance covariance and dcor returns the sample distance correlation. DCOR returns a list with elements

dCov	sample distance covariance
dCor	sample distance correlation
dVarX	distance variance of x sample
dVarY	distance variance of y sample

Note

Two methods of computing the statistics are provided. DCOR is a stand-alone R function that returns a list of statistics. dcov and dcor provide R interfaces to the C implementation, which is usually faster. dcov and dcor call an internal function .dcov.

Note that it is inefficient to compute dCor by:

$$\text{square root of } \text{dcov}(x, y) / \sqrt{\text{dcov}(x, x) * \text{dcov}(y, y)}$$

because the individual calls to dcov involve unnecessary repetition of calculations. For this reason, DCOR computes and returns all four statistics.

Author(s)

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References

Szekely, G.J., Rizzo, M.L., and Bakirov, N.K. (2007), Measuring and Testing Dependence by Correlation of Distances, *Annals of Statistics*, Vol. 35 No. 6, pp. 2769-2794.

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Szekely, G.J. and Rizzo, M.L. (2009), Brownian Distance Covariance, *Annals of Applied Statistics*, Vol. 3, No. 4, 1236-1265.

<http://dx.doi.org/10.1214/09-AOAS312>

Szekely, G.J. and Rizzo, M.L. (2009), Rejoinder: Brownian Distance Covariance, *Annals of Applied Statistics*, Vol. 3, No. 4, 1303-1308.

See Also

[bcdcor](#) [dcovU](#) [pdcor](#) [dcov.test](#) [dcor](#) [ttest](#) [pdcor.test](#)

Examples

```
x <- iris[1:50, 1:4]
y <- iris[51:100, 1:4]
dcov(x, y)
dcov(dist(x), dist(y)) #same thing

## C implementation
dcov(x, y, 1.5)
dcor(x, y, 1.5)

## R implementation
DCOR(x, y, 1.5)
```

edist

E-distance

Description

Returns the E-distances (energy statistics) between clusters.

Usage

```
edist(x, sizes, distance = FALSE, ix = 1:sum(sizes), alpha = 1,
      method = c("cluster", "discoB", "discoF"))
```

Arguments

x	data matrix of pooled sample or Euclidean distances
sizes	vector of sample sizes
distance	logical: if TRUE, x is a distance matrix
ix	a permutation of the row indices of x
alpha	distance exponent in (0,2]
method	how to weight the statistics

Details

A vector containing the pairwise two-sample multivariate \mathcal{E} -statistics for comparing clusters or samples is returned. The e-distance between clusters is computed from the original pooled data, stacked in matrix x where each row is a multivariate observation, or from the distance matrix x of the original data, or distance object returned by `dist`. The first `sizes[1]` rows of the original data matrix are the first sample, the next `sizes[2]` rows are the second sample, etc. The permutation vector `ix` may be used to obtain e-distances corresponding to a clustering solution at a given level in the hierarchy.

The default method `cluster` summarizes the e-distances between clusters in a table. The e-distance between two clusters C_i, C_j of size n_i, n_j proposed by Szekely and Rizzo (2005) is the e-distance $e(C_i, C_j)$, defined by

$$e(C_i, C_j) = \frac{n_i n_j}{n_i + n_j} [2M_{ij} - M_{ii} - M_{jj}],$$

where

$$M_{ij} = \frac{1}{n_i n_j} \sum_{p=1}^{n_i} \sum_{q=1}^{n_j} \|X_{ip} - X_{jq}\|^\alpha,$$

$\|\cdot\|$ denotes Euclidean norm, $\alpha = \text{alpha}$, and X_{ip} denotes the p -th observation in the i -th cluster. The exponent α should be in the interval $(0, 2]$.

The coefficient $\frac{n_i n_j}{n_i + n_j}$ is one-half of the harmonic mean of the sample sizes. The `discoB` and `discoF` methods are related but different ways of summarizing the pairwise differences between samples. The `disco` methods apply the coefficient $\frac{n_i n_j}{2N}$ where N is the total number of observations. This weights each (i, j) statistic by sample size relative to N . See the `disco` topic for more details.

Value

A object of class `dist` containing the lower triangle of the e-distance matrix of cluster distances corresponding to the permutation of indices `ix` is returned. The method attribute of the distance object is assigned a value of type, `index`.

Author(s)

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References

Szekely, G. J. and Rizzo, M. L. (2005) Hierarchical Clustering via Joint Between-Within Distances: Extending Ward's Minimum Variance Method, *Journal of Classification* 22(2) 151-183.
<http://dx.doi.org/10.1007/s00357-005-0012-9>

M. L. Rizzo and G. J. Szekely (2010). DISCO Analysis: A Nonparametric Extension of Analysis of Variance, *Annals of Applied Statistics*, Vol. 4, No. 2, 1034-1055.
<http://dx.doi.org/10.1214/09-AOAS245>

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Szekely, G. J. (2000) Technical Report 03-05, \mathcal{E} -statistics: Energy of Statistical Samples, Department of Mathematics and Statistics, Bowling Green State University.

See Also

[energy.hclust](#) [eqdist](#) [etest](#) [ksample](#) [e](#) [disco](#)

Examples

```
## compute cluster e-distances for 3 samples of iris data
data(iris)
edist(iris[,1:4], c(50,50,50))

## pairwise disco statistics
edist(iris[,1:4], c(50,50,50), method="discoF") #F ratios

## compute e-distances from a distance object
data(iris)
edist(dist(iris[,1:4]), c(50, 50, 50), distance=TRUE, alpha = 1)

## compute e-distances from a distance matrix
data(iris)
d <- as.matrix(dist(iris[,1:4]))
edist(d, c(50, 50, 50), distance=TRUE, alpha = 1)

## compute e-distances from vector of group labels
d <- dist(matrix(rnorm(100), nrow=50))
g <- cutree(energy.hclust(d), k=4)
edist(d, sizes=table(g), ix=rank(g, ties.method="first"))
```

energy.hclust

Hierarchical Clustering by Minimum (Energy) E-distance

Description

Performs hierarchical clustering by minimum (energy) E-distance method.

Usage

```
energy.hclust(dst, alpha = 1)
```

Arguments

dst	dist object
alpha	distance exponent

Details

Dissimilarities are $d(x, y) = \|x - y\|^\alpha$, where the exponent α is in the interval (0,2]. This function performs agglomerative hierarchical clustering. Initially, each of the n singletons is a cluster. At each of $n-1$ steps, the procedure merges the pair of clusters with minimum e-distance. The e-distance between two clusters C_i, C_j of sizes n_i, n_j is given by

$$e(C_i, C_j) = \frac{n_i n_j}{n_i + n_j} [2M_{ij} - M_{ii} - M_{jj}],$$

where

$$M_{ij} = \frac{1}{n_i n_j} \sum_{p=1}^{n_i} \sum_{q=1}^{n_j} \|X_{ip} - X_{jq}\|^\alpha,$$

$\|\cdot\|$ denotes Euclidean norm, and X_{ip} denotes the p -th observation in the i -th cluster.

The return value is an object of class `hclust`, so `hclust` methods such as `print` or `plot` methods, `plclust`, and `cutree` are available. See the documentation for `hclust`.

The e-distance measures both the heterogeneity between clusters and the homogeneity within clusters. \mathcal{E} -clustering ($\alpha = 1$) is particularly effective in high dimension, and is more effective than some standard hierarchical methods when clusters have equal means (see example below). For other advantages see the references.

`edist` computes the energy distances for the result (or any partition) and returns the cluster distances in a `dist` object. See the `edist` examples.

Value

An object of class `hclust` which describes the tree produced by the clustering process. The object is a list with components:

<code>merge</code> :	an $n-1$ by 2 matrix, where row i of <code>merge</code> describes the merging of clusters at step i of the clustering. If an element j in the row is negative, then observation $-j$ was merged at this stage. If j is positive then the merge was with the cluster formed at the (earlier) stage j of the algorithm.
<code>height</code> :	the clustering height: a vector of $n-1$ non-decreasing real numbers (the e-distance between merging clusters)
<code>order</code> :	a vector giving a permutation of the indices of original observations suitable for plotting, in the sense that a cluster plot using this ordering and matrix <code>merge</code> will not have crossings of the branches.
<code>labels</code> :	labels for each of the objects being clustered.
<code>call</code> :	the call which produced the result.
<code>method</code> :	the cluster method that has been used (e-distance).
<code>dist.method</code> :	the distance that has been used to create <code>dst</code> .

Note

Currently `stats::hclust` implements Ward's method by `method="ward.D2"`, which applies the squared distances. That method was previously `"ward"`. Because both `hclust` and `energy` use

the same type of Lance-Williams recursive formula to update cluster distances, now with the additional option `method="ward.D"` in `hclust`, the energy distance method is easily implemented by `energy.hclust`. (Some "Ward" algorithms do not use Lance-Williams, however). Energy clustering (with `alpha=1`) and "ward.D" now return the same result, except that the cluster heights of `energy.hclust` with `alpha=1` are two times the heights from `hclust`.

However, the implementation in the energy package is more than 100 times faster than `hclust`. in a recent benchmark.

Author(s)

Maria L. Rizzo <[mrizzo @ bgsu.edu](mailto:mrizzo@bgsu.edu)> and Gabor J. Szekely

References

Szekely, G. J. and Rizzo, M. L. (2005) Hierarchical Clustering via Joint Between-Within Distances: Extending Ward's Minimum Variance Method, *Journal of Classification* 22(2) 151-183.

<http://dx.doi.org/10.1007/s00357-005-0012-9>

Szekely, G. J. and Rizzo, M. L. (2004) Testing for Equal Distributions in High Dimension, *InterStat*, November (5).

Szekely, G. J. (2000) Technical Report 03-05: \mathcal{E} -statistics: Energy of Statistical Samples, Department of Mathematics and Statistics, Bowling Green State University.

See Also

[edist](#) [ksample.e](#) [eqdist.etest](#) [hclust](#)

Examples

```
## Not run:
library(cluster)
data(animals)
plot(energy.hclust(dist(animals)))

## End(Not run)

data(USArrests)
ecl <- energy.hclust(dist(USArrests))
print(ecl)
plot(ecl)
cutree(ecl, k=3)
cutree(ecl, h=150)

## compare performance of e-clustering, Ward's method, group average method
## when sampled populations have equal means: n=200, d=5, two groups
z <- rbind(matrix(rnorm(1000), nrow=200), matrix(rnorm(1000, 0, 5), nrow=200))
g <- c(rep(1, 200), rep(2, 200))
d <- dist(z)
e <- energy.hclust(d)
a <- hclust(d, method="average")
w <- hclust(d^2, method="ward.D2")
```

```
list("E" = table(cutree(e, k=2) == g), "Ward" = table(cutree(w, k=2) == g),
     "Avg" = table(cutree(a, k=2) == g))
```

 eqdist.etest

Multisample E-statistic (Energy) Test of Equal Distributions

Description

Performs the nonparametric multisample E-statistic (energy) test for equality of multivariate distributions.

Usage

```
eqdist.etest(x, sizes, distance = FALSE,
             method=c("original", "discoB", "discoF"), R)
eqdist.e(x, sizes, distance = FALSE,
         method=c("original", "discoB", "discoF"))
ksample.e(x, sizes, distance = FALSE,
          method=c("original", "discoB", "discoF"), ix = 1:sum(sizes))
```

Arguments

x	data matrix of pooled sample
sizes	vector of sample sizes
distance	logical: if TRUE, first argument is a distance matrix
method	use original (default) or distance components (discoB, discoF)
R	number of bootstrap replicates
ix	a permutation of the row indices of x

Details

The k-sample multivariate \mathcal{E} -test of equal distributions is performed. The statistic is computed from the original pooled samples, stacked in matrix x where each row is a multivariate observation, or the corresponding distance matrix. The first $\text{sizes}[1]$ rows of x are the first sample, the next $\text{sizes}[2]$ rows of x are the second sample, etc.

The test is implemented by nonparametric bootstrap, an approximate permutation test with R replicates.

The function `eqdist.e` returns the test statistic only; it simply passes the arguments through to `eqdist.etest` with $R = 0$.

The k-sample multivariate \mathcal{E} -statistic for testing equal distributions is returned. The statistic is computed from the original pooled samples, stacked in matrix x where each row is a multivariate observation, or from the distance matrix x of the original data. The first $\text{sizes}[1]$ rows of x are the first sample, the next $\text{sizes}[2]$ rows of x are the second sample, etc.

The two-sample \mathcal{E} -statistic proposed by Szekely and Rizzo (2004) is the e-distance $e(S_i, S_j)$, defined for two samples S_i, S_j of size n_i, n_j by

$$e(S_i, S_j) = \frac{n_i n_j}{n_i + n_j} [2M_{ij} - M_{ii} - M_{jj}],$$

where

$$M_{ij} = \frac{1}{n_i n_j} \sum_{p=1}^{n_i} \sum_{q=1}^{n_j} \|X_{ip} - X_{jq}\|,$$

$\|\cdot\|$ denotes Euclidean norm, and X_{ip} denotes the p -th observation in the i -th sample.

The original (default method) k -sample \mathcal{E} -statistic is defined by summing the pairwise e-distances over all $k(k-1)/2$ pairs of samples:

$$\mathcal{E} = \sum_{1 \leq i < j \leq k} e(S_i, S_j).$$

Large values of \mathcal{E} are significant.

The discoB method computes the between-sample disco statistic. For a one-way analysis, it is related to the original statistic as follows. In the above equation, the weights $\frac{n_i n_j}{n_i + n_j}$ are replaced with

$$\frac{n_i + n_j}{2N} \frac{n_i n_j}{n_i + n_j} = \frac{n_i n_j}{2N}$$

where N is the total number of observations: $N = n_1 + \dots + n_k$.

The discoF method is based on the disco F ratio, while the discoB method is based on the between sample component.

Also see `disco` and `disco.between` functions.

Value

A list with class `htest` containing

<code>method</code>	description of test
<code>statistic</code>	observed value of the test statistic
<code>p.value</code>	approximate p-value of the test
<code>data.name</code>	description of data

`eqdist.e` returns test statistic only.

Note

The pairwise e-distances between samples can be conveniently computed by the `edist` function, which returns a `dist` object.

Author(s)

Maria L. Rizzo <mrizzo@bgsu.edu> and Gabor J. Szekely

References

Szekely, G. J. and Rizzo, M. L. (2004) Testing for Equal Distributions in High Dimension, *InterStat*, November (5).

M. L. Rizzo and G. J. Szekely (2010). DISCO Analysis: A Nonparametric Extension of Analysis of Variance, *Annals of Applied Statistics*, Vol. 4, No. 2, 1034-1055.

<http://dx.doi.org/10.1214/09-AOAS245>

Szekely, G. J. (2000) Technical Report 03-05: \mathcal{E} -statistics: Energy of Statistical Samples, Department of Mathematics and Statistics, Bowling Green State University.

See Also

[ksample.e](#), [edist](#), [disco](#), [disco.between](#), [energy.hclust](#).

Examples

```
data(iris)

## test if the 3 varieties of iris data (d=4) have equal distributions
eqdist.etest(iris[,1:4], c(50,50,50), R = 199)

## example that uses method="disco"
x <- matrix(rnorm(100), nrow=20)
y <- matrix(rnorm(100), nrow=20)
X <- rbind(x, y)
d <- dist(X)

# should match edist default statistic
set.seed(1234)
eqdist.etest(d, sizes=c(20, 20), distance=TRUE, R = 199)

# comparison with edist
edist(d, sizes=c(20, 10), distance=TRUE)

# for comparison
g <- as.factor(rep(1:2, c(20, 20)))
set.seed(1234)
disco(d, factors=g, distance=TRUE, R=199)

# should match statistic in edist method="discoB", above
set.seed(1234)
disco.between(d, factors=g, distance=TRUE, R=199)
```

Description

Deprecated: use `indep.test` with `method = mvI`. Computes a multivariate nonparametric E-statistic and test of independence.

Usage

```
indep.e(x, y)
indep.etest(x, y, R)
```

Arguments

x	matrix: first sample, observations in rows
y	matrix: second sample, observations in rows
R	number of replicates

Details

Computes the coefficient \mathcal{I} and performs a nonparametric \mathcal{E} -test of independence. The test decision is obtained via bootstrap, with R replicates. The sample sizes (number of rows) of the two samples must agree, and samples must not contain missing values. The statistic $\mathcal{E} = n\mathcal{I}^2$ is a ratio of V-statistics based on interpoint distances $\|x_i - y_j\|$. See the reference below for details.

Value

The sample coefficient \mathcal{I} is returned by `indep.e`. The function `indep.etest` returns a list with class `htest` containing

method	description of test
statistic	observed value of the coefficient \mathcal{I}
p.value	approximate p-value of the test
data.name	description of data

Author(s)

Maria L. Rizzo <mrizzo@bgsu.edu> and Gabor J. Szekely

References

Bakirov, N.K., Rizzo, M.L., and Szekely, G.J. (2006), A Multivariate Nonparametric Test of Independence, *Journal of Multivariate Analysis* 93/1, 58-80,
<http://dx.doi.org/10.1016/j.jmva.2005.10.005>

Examples

```
## Not run:
## independent univariate data
x <- sin(runif(30, 0, 2*pi) * 2)
y <- sin(runif(30, 0, 2*pi) * 4)
indep.etest(x, y, R = 99)

## dependent multivariate data
Sigma <- matrix(c(1, .1, 0, 0, 1, 0, 0, .1, 1), 3, 3)
x <- mvrnorm(30, c(0, 0, 0), diag(3))
y <- mvrnorm(30, c(0, 0, 0), Sigma) * x
indep.etest(x, y, R = 99)

## End(Not run)
```

indep.test

Energy-tests of Independence

Description

Computes a multivariate nonparametric test of independence. The default method implements the distance covariance test [dcov.test](#).

Usage

```
indep.test(x, y, method = c("dcov", "mVI"), index = 1, R)
```

Arguments

x	matrix: first sample, observations in rows
y	matrix: second sample, observations in rows
method	a character string giving the name of the test
index	exponent on Euclidean distances
R	number of replicates

Details

`indep.test` with the default `method = "dcov"` computes the distance covariance test of independence. `index` is an exponent on the Euclidean distances. Valid choices for `index` are in $(0, 2]$, with default value 1 (Euclidean distance). The arguments are passed to the `dcov.test` function. See the help topic [dcov.test](#) for the description and documentation and also see the references below.

`indep.test` with `method = "mVI"` computes the coefficient \mathcal{I}_n and performs a nonparametric \mathcal{E} -test of independence. The arguments are passed to `mVI.test`. The `index` argument is ignored (`index = 1` is applied). See the help topic [mVI.test](#) and also see the reference (2006) below for details.

The test decision is obtained via bootstrap, with R replicates. The sample sizes (number of rows) of the two samples must agree, and samples must not contain missing values.

These energy tests of independence are based on related theoretical results, but different test statistics. The dcov method is faster than mvI method by approximately a factor of $O(n)$.

Value

indep.test returns a list with class htest containing

method	description of test
statistic	observed value of the test statistic $n\mathcal{V}_n^2$ or $n\mathcal{I}_n^2$
estimate	\mathcal{V}_n or \mathcal{I}_n
estimates	a vector [dCov(x,y), dCor(x,y), dVar(x), dVar(y)] (method dcov)
replicates	replicates of the test statistic
p.value	approximate p-value of the test
data.name	description of data

Note

As of energy-1.1-0, indep.etest is deprecated and replaced by indep.test, which has methods for two different energy tests of independence. indep.test applies the distance covariance test (see dcov.test) by default (method = "dcov"). The original indep.etest applied the independence coefficient \mathcal{I}_n , which is now obtained by method = "mvI".

Author(s)

Maria L. Rizzo <mrizzo @ bgsu.edu> and Gabor J. Szekely

References

Szekely, G.J. and Rizzo, M.L. (2009), Brownian Distance Covariance, *Annals of Applied Statistics*, Vol. 3 No. 4, pp. 1236-1265. (Also see discussion and rejoinder.)

<http://dx.doi.org/10.1214/09-AOAS312>

Szekely, G.J., Rizzo, M.L., and Bakirov, N.K. (2007), Measuring and Testing Dependence by Correlation of Distances, *Annals of Statistics*, Vol. 35 No. 6, pp. 2769-2794.

<http://dx.doi.org/10.1214/009053607000000505>

Bakirov, N.K., Rizzo, M.L., and Szekely, G.J. (2006), A Multivariate Nonparametric Test of Independence, *Journal of Multivariate Analysis* 93/1, 58-80,

<http://dx.doi.org/10.1016/j.jmva.2005.10.005>

See Also

[dcov.test](#) [mvI.test](#) [dcov](#) [mvI](#)

Examples

```

## independent multivariate data
x <- matrix(rnorm(60), nrow=20, ncol=3)
y <- matrix(rnorm(40), nrow=20, ncol=2)
indep.test(x, y, method = "dcov", R = 99)
indep.test(x, y, method = "mvI", R = 99)

## Not run:
## dependent multivariate data
if (require(MASS)) {
  Sigma <- matrix(c(1, .1, 0, 0, 1, 0, 0, .1, 1), 3, 3)
  x <- mvrnorm(30, c(0, 0, 0), diag(3))
  y <- mvrnorm(30, c(0, 0, 0), Sigma) * x
  indep.test(x, y, R = 99) #dcov method
  indep.test(x, y, method = "mvI", R = 99)
}

## End(Not run)

## Not run:
## compare the computing time
x <- mvrnorm(50, c(0, 0, 0), diag(3))
y <- mvrnorm(50, c(0, 0, 0), Sigma) * x
set.seed(123)
system.time(indep.test(x, y, method = "dcov", R = 1000))
set.seed(123)
system.time(indep.test(x, y, method = "mvI", R = 1000))

## End(Not run)

```

mvI.test

Energy Statistic Test of Independence

Description

Computes the multivariate nonparametric E-statistic and test of independence based on independence coefficient \mathcal{I}_n .

Usage

```

mvI.test(x, y, R)
mvI(x, y)

```

Arguments

x	matrix: first sample, observations in rows
y	matrix: second sample, observations in rows
R	number of replicates

Details

Computes the coefficient \mathcal{I} and performs a nonparametric \mathcal{E} -test of independence. The test decision is obtained via bootstrap, with R replicates. The sample sizes (number of rows) of the two samples must agree, and samples must not contain missing values. The statistic $\mathcal{E} = n\mathcal{I}^2$ is a ratio of V-statistics based on interpoint distances $\|x_i - y_j\|$. See the reference below for details.

Value

mvI returns the statistic. mvI.test returns a list with class htest containing

method	description of test
statistic	observed value of the test statistic $n\mathcal{I}_n^2$
estimate	\mathcal{I}_n
replicates	replicates of the test statistic
p.value	approximate p-value of the test
data.name	description of data

Note

Historically this is the first energy test of independence. The distance covariance test `dcov.test`, distance correlation `dcor`, and related methods are more recent (2007,2009). The distance covariance test is faster and has different properties than `mvI.test`. Both methods are based on a population independence coefficient that characterizes independence and both tests are statistically consistent.

Author(s)

Maria L. Rizzo <mrizzo @ bgsu.edu> and Gabor J. Szekely

References

Bakirov, N.K., Rizzo, M.L., and Szekely, G.J. (2006), A Multivariate Nonparametric Test of Independence, *Journal of Multivariate Analysis* 93/1, 58-80,
<http://dx.doi.org/10.1016/j.jmva.2005.10.005>

See Also

[indep.test](#) [mvI.test](#) [dcov.test](#) [dcov](#)

mvnorm.etest *E-statistic (Energy) Test of Multivariate Normality*

Description

Performs the E-statistic (energy) test of multivariate or univariate normality.

Usage

```
mvnorm.etest(x, R)
mvnorm.e(x)
normal.e(x)
```

Arguments

x data matrix of multivariate sample, or univariate data vector
 R number of bootstrap replicates

Details

If x is a matrix, each row is a multivariate observation. The data will be standardized to zero mean and identity covariance matrix using the sample mean vector and sample covariance matrix. If x is a vector, the univariate statistic normal.e(x) is returned. If the data contains missing values or the sample covariance matrix is singular, NA is returned.

The \mathcal{E} -test of multivariate normality was proposed and implemented by Szekely and Rizzo (2005). The test statistic for d-variate normality is given by

$$\mathcal{E} = n \left(\frac{2}{n} \sum_{i=1}^n E \|y_i - Z\| - E \|Z - Z'\| - \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n \|y_i - y_j\| \right),$$

where y_1, \dots, y_n is the standardized sample, Z, Z' are iid standard d-variate normal, and $\|\cdot\|$ denotes Euclidean norm.

The \mathcal{E} -test of multivariate (univariate) normality is implemented by parametric bootstrap with R replicates.

If R=0 the summary for the test gives the test statistic only (no p-value).

Value

The value of the \mathcal{E} -statistic for univariate normality is returned by normal.e. The value of the \mathcal{E} -statistic for multivariate normality is returned by mvnorm.e.

mvnorm.etest returns a list with class htest containing

method	description of test
statistic	observed value of the test statistic
p.value	approximate p-value of the test
data.name	description of data

Author(s)

Maria L. Rizzo <mrizzo @ bgsu.edu> and Gabor J. Szekely

References

Szekely, G. J. and Rizzo, M. L. (2005) A New Test for Multivariate Normality, *Journal of Multivariate Analysis*, 93/1, 58-80, <http://dx.doi.org/10.1016/j.jmva.2003.12.002>.

Rizzo, M. L. (2002). A New Rotation Invariant Goodness-of-Fit Test, Ph.D. dissertation, Bowling Green State University.

Szekely, G. J. (1989) Potential and Kinetic Energy in Statistics, Lecture Notes, Budapest Institute of Technology (Technical University).

Examples

```
## compute normality test statistics for iris Setosa data
data(iris)
mvnorm.e(iris[1:50, 1:4])
normal.e(iris[1:50, 1])

## test if the iris Setosa data has multivariate normal distribution
mvnorm.etest(iris[1:50,1:4], R = 199)

## test a univariate sample for normality
x <- runif(50, 0, 10)
mvnorm.etest(x, R = 199)
```

pdcor

Partial distance correlation and covariance

Description

Partial distance correlation pdcor, pdcov, and tests.

Usage

```
pdcov.test(x, y, z, R)
pdcor.test(x, y, z, R)
pdcor(x, y, z)
pdcov(x, y, z)
```

Arguments

x	data matrix or dist object of first sample
y	data matrix or dist object of second sample
z	data matrix or dist object of third sample
R	replicates for permutation test

Details

`pdcor(x, y, z)` and `pdcov(x, y, z)` compute the partial distance correlation and partial distance covariance, respectively, of x and y removing z .

A test for zero partial distance correlation (or zero partial distance covariance) is implemented in `pdcor.test`, and `pdcov.test`.

If the argument is a matrix, it is treated as a data matrix and distances are computed (observations in rows). If the arguments are distances or dissimilarities, they must be distance (`dist`) objects. For symmetric, zero-diagonal dissimilarity matrices, use `as.dist` to convert to a `dist` object.

Value

Each test returns an object of class `htest`.

Author(s)

Maria L. Rizzo <`mrizzo @ bgsu.edu`> and Gabor J. Szekely

References

Szekely, G.J. and Rizzo, M.L. (2014), Partial Distance Correlation with Methods for Dissimilarities. *Annals of Statistics*, Vol. 42 No. 6, 2382-2412.

Examples

```
n = 30
R <- 199

## mutually independent standard normal vectors
x <- rnorm(n)
y <- rnorm(n)
z <- rnorm(n)

pdcor(x, y, z)
pdcov(x, y, z)
pdcov.test(x, y, z, R=R)
print(pdcor.test(x, y, z, R=R))

if (require(MASS)) {
  p = 4
  mu <- rep(0, p)
  Sigma <- diag(p)

  ## linear dependence
  y <- mvrnorm(n, mu, Sigma) + x
  print(pdcov.test(x, y, z, R=R))

  ## non-linear dependence
  y <- mvrnorm(n, mu, Sigma) * x
  print(pdcov.test(x, y, z, R=R))
}
```

poisson.mtest

*Mean Distance Test for Poisson Distribution***Description**

Performs the mean distance goodness-of-fit test of Poisson distribution with unknown parameter.

Usage

```
poisson.mtest(x, R)
poisson.m(x)
```

Arguments

x	vector of nonnegative integers, the sample data
R	number of bootstrap replicates

Details

The mean distance test of Poissonity was proposed and implemented by Szekely and Rizzo (2004). The test is based on the result that the sequence of expected values $E|X-j|$, $j=0,1,2,\dots$ characterizes the distribution of the random variable X . As an application of this characterization one can get an estimator $\hat{F}(j)$ of the CDF. The test statistic (see [poisson.m](#)) is a Cramer-von Mises type of distance, with M-estimates replacing the usual EDF estimates of the CDF:

$$M_n = n \sum_{j=0}^{\infty} (\hat{F}(j) - F(j; \hat{\lambda}))^2 f(j; \hat{\lambda}).$$

The test is implemented by parametric bootstrap with R replicates.

Value

The function `poisson.m` returns the test statistic. The function `poisson.mtest` returns a list with class `htest` containing

method	Description of test
statistic	observed value of the test statistic
p.value	approximate p-value of the test
data.name	description of data
estimate	sample mean

Author(s)

Maria L. Rizzo <mrizzo@bgsu.edu> and Gabor J. Szekely

References

Szekely, G. J. and Rizzo, M. L. (2004) Mean Distance Test of Poisson Distribution, *Statistics and Probability Letters*, 67/3, 241-247. <http://dx.doi.org/10.1016/j.spl.2004.01.005>.

Examples

```
x <- rpois(20, 1)
poisson.m(x)
poisson.mtest(x, R = 199)
```

Unbiased distance covariance

Unbiased dcov and bias-corrected dcor statistics

Description

These functions compute unbiased estimators of squared distance covariance and a bias-corrected estimator of (squared) distance correlation.

Usage

```
bcdcor(x, y)
dcovU(x, y)
```

Arguments

x	data or dist object of first sample
y	data or dist object of second sample

Details

The unbiased (squared) dcov is inner product definition of dCov, in the Hilbert space of U-centered distance matrices.

The sample sizes (number of rows) of the two samples must agree, and samples must not contain missing values. Arguments x, y can optionally be `dist` objects; otherwise these arguments are treated as data.

Value

dcovU returns the unbiased estimator of squared dcov. bcdcor returns a bias-corrected estimator of squared dcor.

Note

Unbiased distance covariance (SR2014) corresponds to the biased (original) $dCov^2$. Since $dcovU$ is an unbiased statistic, it is signed and we do not take the square root. For the original distance covariance test of independence (SRB2007, SR2009), the distance covariance test statistic is the V-statistic $n dCov^2 = n\mathcal{V}_n^2$ (not $dCov$). Similarly, $bcdcor$ is bias-corrected, so we do not take the square root as with $dCor$.

Author(s)

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References

Szekely, G.J. and Rizzo, M.L. (2014), Partial Distance Correlation with Methods for Dissimilarities, *Annals of Statistics*, Vol. 42 No. 6, 2382-2412.

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<http://dx.doi.org/10.1214/009053607000000505>

Szekely, G.J. and Rizzo, M.L. (2009), Brownian Distance Covariance, *Annals of Applied Statistics*, Vol. 3, No. 4, 1236-1265.

<http://dx.doi.org/10.1214/09-AOAS312>

Examples

```
x <- iris[1:50, 1:4]
y <- iris[51:100, 1:4]
dcovU(x, y)
bcdcor(x, y)
```

U_product

Inner product in the Hilbert space of U-centered distance matrices

Description

Stand-alone function to compute the inner product in the Hilbert space of U-centered distance matrices, as in the definition of partial distance covariance.

Usage

```
U_product(U, V)
```

Arguments

U	U-centered distance matrix
V	U-centered distance matrix

Details

Note that `pdcor`, etc. functions include the centering and projection operations, so that these stand alone versions are not needed except in case one wants to check the internal computations.

Exported from `U_product.cpp`.

Value

`U_product` returns the inner product, a scalar.

Author(s)

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References

Szekely, G.J. and Rizzo, M.L. (2014), Partial Distance Correlation with Methods for Dissimilarities, *Annals of Statistics*, Vol. 42, No. 6, pp. 2382-2412. DOI [dx.doi.org/10.1214/14-AOS1255](https://doi.org/10.1214/14-AOS1255)
<http://projecteuclid.org/euclid.aos/1413810731>

Examples

```
x <- iris[1:10, 1:4]
y <- iris[11:20, 1:4]
M1 <- as.matrix(dist(x))
M2 <- as.matrix(dist(y))
U <- U_center(M1)
V <- U_center(M2)

U_product(U, V)
dcovU_stats(M1, M2)
```

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