Examples from Multilevel Software
Comparative Reviews

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Abstract

The Center for Multilevel Modelling at the Institute of Education, London maintains a web site of “Software reviews of multilevel modeling packages”. The data sets discussed in the reviews are available at this web site. We have incorporated these data sets in the mlmRev package for R and, in this vignette, provide the results of fitting several models to these data sets.

1 Introduction

R is an Open Source implementation of John Chambers’ S language for data analysis and graphics. R was initially developed by Ross Ihaka and Robert Gentleman of the University of Auckland and now is developed and maintained by an international group of statistical computing experts.

In addition to being Open Source software, which means that anyone can examine the source code to see exactly how the computations are being carried out, R is freely available from a network of archive sites on the Internet. There are precompiled versions for installation on the Windows operating system, Mac OS X and several distributions of the Linux operating system. Because the source code is available those interested in doing so can compile their own version if they wish.

R provides an environment for interactive computing with data and for graphical display of data. Users and developers can extend the capabilities of
R by writing their own functions in the language and by creating packages of functions and data sets. Many such packages are available on the archive network called CRAN (Comprehensive R Archive Network) for which the parent site is [http://cran.r-project.org](http://cran.r-project.org). One such package called lme4 (along with a companion package called Matrix) provides functions to fit and display linear mixed models and generalized linear mixed models, which are the statisticians’ names for the models called multilevel models or hierarchical linear models in other disciplines. The lattice package provides functions to generate several high level graphics plots that help with the visualization of the types of data to which such models are fit. Finally, the mlmRev package provides the data sets used in the “Software Reviews of Multilevel Modeling Packages” from the Multilevel Modeling Group at the Institute of Education in the UK. This package also contains several other data sets from the multilevel modeling literature.

The software reviews mentioned above were intended to provide comparison of the speed and accuracy of many different packages for fitting multilevel models. As such, there is a standard set of models that fit to each of the data sets in each of the packages that were capable of doing the fit. We will fit these models for comparative purposes but we will also do some graphical exploration of the data and, in some cases, discuss alternative models.

We follow the general outline of the previous reviews, beginning with simpler structures and moving to the more complex structures. Because the previous reviews were performed on older and considerably slower computers than the ones on which this vignette will be compiled, the timings produced by the system.time function and shown in the text should not be compared with previous timings given on the web site. They are an indication of the times required to fit such models to these data sets on recent computers with processors running at around 2 GHz or faster.

## 2 Two-level normal models

In the multilevel modeling literature a two-level model is one with two levels of random variation; the per-observation noise term and random effects which are grouped according to the levels of a factor. We call this factor a grouping factor. If the response is measured on a continuous scale (more or less) our initial models are based on a normal distribution for the per-observation noise and for the random effects. Thus such a model is called a “two-level normal
model” even though it has only one grouping factor for the random effects.

2.1 The Exam data

The data set called Exam provides the normalized exam scores attained by 4,059 students from 65 schools in inner London. Some of the covariates available with this exam score are the school the student attended, the sex of the student, the school gender (boys, girls, or mixed) and the student’s result on the Standardised London Reading test.

The R functions `str` and `summary` can be used to examine the structure of a data set (or, in general, any R object) and to provide a summary of an object.

```r
> str(Exam)
'data.frame': 4059 obs. of 10 variables:
$ school : Factor w/ 65 levels "1","2","3","4",..: 1 1 1 1 1 1 1 1 1 1 ... 
$ normexam: num 0.261 0.134 -1.724 0.968 0.544 ... 
$ schgend : Factor w/ 3 levels "mixed","boys",..: 1 1 1 1 1 1 1 1 1 1 ... 
$ schavg : num 0.166 0.166 0.166 0.166 0.166 ... 
$ vr : Factor w/ 3 levels "bottom 25%","mid 50%",..: 2 2 2 2 2 2 2 2 2 2 ... 
$ intake : Factor w/ 3 levels "bottom 25%","mid 50%",..: 1 2 3 2 2 1 3 2 2 3 ... 
$ standLRT: num 0.619 0.206 -1.365 0.206 0.371 ... 
$ sex : Factor w/ 2 levels "F","M": 1 1 2 1 1 2 2 1 2 ... 
$ type : Factor w/ 2 levels "Mxd","Sngl": 1 1 1 1 1 1 1 1 1 1 ... 
$ student : Factor w/ 650 levels "1","2","3","4",..: 143 145 142 141 138 155 158 115 117 113 ... 

> summary(Exam)
school normexam schgend schavg
14 :198 Min. :-3.666072 mixed:2169 Min. :-0.75596
17 :126 1st Qu.:-0.699505 boys :513 1st Qu.:-0.14934
18 :120 Median : 0.004322 girls:1377 Median :-0.02020
49 :113 Mean :-0.000114 Mean : 0.00181
8 : 102 3rd Qu.: 0.678759 Mean : 0.00181
15 :  91 Max. : 3.666091 3rd Qu.: 0.61906
(Other):3309

vr intake standLRT sex type
bottom 25%: 640 bottom 25%:1176 Min. :-2.93495 F:2436 Mxd :2169
mid 50% :2263 mid 50% :2344 1st Qu.:-0.62071 M:1623 Sngl:1890
top 25% :1156 top 25% : 539 Median : 0.04050
Mean : 0.00181
3rd Qu.: 0.61906
Max. : 3.01595
(Other):3859

student
20 : 34
54 : 34
15 : 33
22 : 33
31 : 33
59 : 33
(Other):3859
2.2 Model fits and timings

The first model to fit to the Exam data incorporates fixed-effects terms for the pretest score, the student’s sex and the school gender. The only random-effects term is an additive shift associated with the school.

```r
> (Em1 <- lmer(normexam ~ standLRT + sex + schgend + (1|school), Exam))
Linear mixed model fit by REML ['lmerMod']
Formula: normexam ~ standLRT + sex + schgend + (1 | school)
Data: Exam
REML criterion at convergence: 9347.674
Random effects:
  Groups   Name        Std.Dev.
  school (Intercept) 0.293
  Residual          0.750
Number of obs: 4059, groups: school, 65
Fixed Effects:
                (Intercept)       standLRT       sexM  schgendboys  schgendgirls
          -0.001049        0.559754       -0.167392        0.177691        0.158997
```

The `system.time` function can be used to time the execution of an R expression. It returns a vector of five numbers giving the user time (time spend executing applications code), the system time (time spent executing system functions called by the applications code), the elapsed time, and the user and system time for any child processes. The first number is what is commonly viewed as the time required to do the model fit. (The elapsed time is unsuitable because it can be affected by other processes running on the computer.) These times are in seconds. On modern computers this fit takes only a fraction of a second.

```r
> system.time(lmer(normexam ~ standLRT + sex + schgend + (1|school), Exam))
user  system elapsed
  0.03   0.00   0.03
```

2.3 Interpreting the fit

As can be seen from the output, the default method of fitting a linear mixed model is restricted maximum likelihood (REML). The estimates of the variance components correspond to those reported by other packages as given on the Multilevel Modelling Group’s web site. Note that the estimates of the variance components are given on the scale of the variance and on the scale of the standard deviation. That is, the values in the column headed `Std.Dev.` are simply the square roots of the corresponding entry in the `Variance` column. They are **not** standard errors of the estimate of the variance.
The estimates of the fixed-effects are different from those quoted on the web site because the terms for \texttt{sex} and \texttt{schgend} use a different parameterization than in the reviews. Here the reference level of \texttt{sex} is female (F) and the coefficient labelled \texttt{sexM} represents the difference for males compared to females. Similarly the reference level of \texttt{schgend} is mixed and the two coefficients represent the change from mixed to boys only and the change from mixed to girls only. The value of the coefficient labelled \texttt{Intercept} is affected by both these changes as is the value of the REML criterion.

To reproduce the results obtained from other packages, we must change the reference level for each of these factors.

```r
> Exam$sex <- relevel(Exam$sex, "M")
> Exam$schgend <- relevel(Exam$schgend, "girls")
> (Em2 <- lmer(normexam ~ standLRT + sex + schgend + (1|school), Exam))
```

\begin{verbatim}
Linear mixed model fit by REML ['lmerMod']
Formula: normexam ~ standLRT + sex + schgend + (1|school)
 Data: Exam
REML criterion at convergence: 9347.674
Random effects:
 Groups     Name     Std.Dev.
school  (Intercept) 0.293
Residual            0.750
Number of obs: 4059, groups: school, 65
Fixed Effects:
(Intercept)  standLRT  sexF  schgendmixed  schgendboys
-0.009444   0.559754   0.167392  -0.158997   0.018694
\end{verbatim}

The coefficients now correspond to those in the tables on the web site. It happens that the REML criterion at the optimum in this fit is the same as in the previous fit, but you cannot depend on this occurring. In general the value of the REML criterion at the optimum depends on the parameterization used for the fixed effects.

### 2.4 Further exploration

#### 2.4.1 Checking consistency of the data

It is important to check the consistency of data before trying to fit sophisticated models. One should plot the data in many different ways to see if it looks reasonable and also check relationships between variables.

For example, each observation in these data is associated with a particular student. The variable \texttt{student} is not a unique identifier of the student as it only has 650 unique values. It is intended to be a unique identifier within a school but it is not. To show this we create a factor that is the interaction of school and student then drop unused levels.
Notice that there are 4059 observations but only 4055 unique levels of student within school. We can check the ones that are duplicated

```r
> as.character(Exam$ids[which(duplicated(Exam$ids))])
[1] "43:86" "50:39" "52:2" "52:21"
```

One of these cases

```r
> subset(Exam, ids == '43:86')
school normexam schgend schavg vr intake standLRT sex type
2758 43 -0.8526700 mixed 0.4334322 top 25% mid 50% 0.1231502 M Mxd
2759 43 0.8219882 mixed 0.4334322 top 25% top 25% -0.0421520 F Mxd

> xtabs(~ sex + school, Exam, subset = school %in% c(43, 50, 52), drop = TRUE)
school sex 43 50 52
M 1 35 61
F 60 38 0
```

is particularly interesting. Notice that one of the students numbered 86 in school 43 is the only male student out of 61 students from this school who took the exam. It is quite likely that this student’s score was attributed to the wrong school and that the school is in fact a girls-only school, not a mixed-sex school.

The causes of the other three cases of duplicate student numbers within a school are not as clear. It would be necessary to go back to the original data records to check these.

The cross-tabulation of the students by sex and school for the mixed-sex schools

```r
> xtabs(~ sex + school, Exam, subset = type == "Mxd", drop = TRUE)
```
shows another anomaly. School 47 is similar to school 43 in that, although it is classified as a mixed-sex school, 81 male students and only one female student took the exam. It is likely that the school was misrecorded for this one female student and the school is a male-only school.

Another school is worth noting. There were only eight students from school 54 who took the exam so any within-school estimates from this school will be unreliable.

A mosaic plot (Figure 1) produced with

```r
> ExamMxd <- within(subset(Exam, type == "Mxd"), school <- factor(school))
> mosaicplot(~ school + sex, ExamMxd)
```

helps to detect mixed-sex schools with unusually large or unusually small ratios of females to males taking the exam.

### 2.4.2 Preliminary graphical displays

In addition to the pretest score (standLRT), the predictor variables used in this model are the student's sex and the school gender, which is coded as having three levels. There is some redundancy in these two variables in that all the students in a boys-only school must be male. For graphical exploration we convert from schgend to type, an indicator of whether the school is a mixed-sex school or a single-sex school, and plot the response versus the pretest score for each combination of sex and school type.

This plot is created with the `xyplot` from the `lattice` package as (essentially)

```r
> xyplot(normexam ~ standLRT | sex * type, Exam, type = c("g", "p", "smooth"))
```

The formula would be read as “plot normexam by standLRT given sex by (school) type”. A few other arguments were added in the actual call to make the axis annotations more readable.

Figure 2 shows the even after accounting for a student's sex, pretest score and school type, there is considerable variation in the response. We may
Figure 1: A mosaic plot of the sex distribution by school. The areas of the rectangles are proportional to the number of students of that sex from that school who took the exam. Schools with an unusually large or unusually small ratio of females to males are highlighted.
Figure 2: Normalized exam score versus pretest (Standardized London Reading Test) score for 4095 students from 65 schools in inner London. The panels on the left show the male students' scores; those on the right show the females' scores. The top row of panels shows the scores of students in single-sex schools and the bottom row shows the scores of students in mixed-sex schools. A scatterplot smoother line for each panel has been added to help visualize the trend.
attribute some of this variation to differences in schools but the fitted model indicates that most of the variation is unaccounted or “residual” variation.

In some ways the high level of residual variation obscures the pattern in the data. By removing the data points and overlaying the scatterplot smoothers we can concentrate on the relationships between the covariates. The call to `xyplot` is essentially

```r
> xyplot(normexam ~ standLRT, Exam, groups = sex:type, type = c("g", "smooth"))
```

Figure 3 is a remarkable plot in that it shows nearly a perfect “main effects” relationship of the response with the three covariates and almost no interaction. It is rare to see real data follow a simple theoretical relationship so closely.

To elaborate, we can see that for each of the four groups the smoothed relationship between the exam score and the pretest score is close to a straight line and that the lines are close to being parallel. The only substantial deviation is in the smoothed relationship for the males in single-sex schools and this is the group with the fewest observations and hence the least precision in the estimated relationship. The lack of parallelism for this group is most apparent in the region of extremely low pretest scores where there are few observations and a single student who had a low pretest score and a moderate
post-test score can substantially influence the curve. Five or six such points can be seen in the upper left panel of Figure 2.

We should also notice the ordering of the lines and the spacing between the lines. The smoothed relationships for students in single-sex schools are consistently above those in the mixed-sex schools and, except for the region of low pretest scores described above, the relationship for the females in a given type of school is consistently above that for the males. Furthermore the distance between the female and male lines in the single-sex schools is approximately the same as the corresponding distance in the mixed-sex schools. We would summarize this by saying that there is a positive effect for females versus males and a positive effect for single-sex versus mixed-sex and no indication of interaction between these factors.

2.4.3 The effect of schools

We can check for patterns within and between schools by plotting the response versus the pretest by school. Because there appear to be differences in this relationship for single-sex versus mixed-sex schools and for females versus males we consider these separately.

In Figure 4 we plot the normalized exam scores versus the pretest score by school for female students in single-sex schools. The plot is produced as

\[ xyplot(\text{normexam} \sim \text{standLRT} | \text{school}, \text{Exam}, \text{type} = \text{c("g", "p", "r")}, \text{subset} = \text{sex} == \text{"F"} \& \text{type} == \text{"Sngl"}) \]

The "r" in the `type` argument adds a simple linear regression line to each panel.

The first thing we notice in Figure 4 is that school 48 is an anomaly because only two students in this school took the exam. Because within-school results based on only two students are unreliable, we will exclude this school from further plots (but we do include these data when fitting comprehensive models).

Although the regression lines on the panels can help us to look for variation in the schools, the ordering of the panels is, for our purposes, random. We recreate this plot in Figure 5 using

\[ xyplot(\text{normexam} \sim \text{standLRT} | \text{school}, \text{Exam}, \text{type} = \text{c("g", "p", "r")}, \text{subset} = \text{sex} == \text{"F"} \& \text{type} == \text{"Sngl"} \& \text{school} != 48, \text{index.cond} = \text{function(x, y) coef(lm(y ~ x))[1]}) \]
Figure 4: Normalized exam scores versus pretest (Standardized London Reading Test) score by school for female students in single-sex schools.
Figure 5: Normalized exam scores versus pretest (Standardized London Reading Test) score by school for female students in single-sex schools. School 48 where only two students took the exam has been eliminated and the panels have been ordered by increasing intercept (predicted normalized score for a pretest score of 0) of the regression line.
so that the panels are ordered (from left to right starting at the bottom row) by increasing intercept for the regression line (i.e. by increasing predicted exam score for a student with a pretest score of 0).

Alternatively, we could order the panels by increasing slope of the within-school regression lines, as in Figure 6.

Although it is informative to plot the within-school regression lines we need to assess the variability in the estimates of the coefficients before concluding if there is “significant” variability between schools. We can obtain the individual regression fits with the \texttt{lmList} function

\begin{verbatim}
> show(ExamFS <- lmList(normexam ~ standLRT | school, Exam, + subset = sex == "F" & type == "Sngl" & school != 48))

Call: lmList(formula = normexam ~ standLRT | school, data = Exam, subset = sex == "F" & type == "Sngl", + study = exam)

Coefficients:

(Intercept) standLRT
2 0.48227991 0.7612884
6 0.60321439 0.5353444
7 0.39852689 0.2422785
8 -0.02519463 0.5674053
16 -0.38564292 0.4069399
18 -0.05733995 0.3593830
21 0.26872018 0.5544939
25 -0.26779146 0.5320575
29 0.20442314 0.4005158
30 0.11885028 0.8059021
31 -0.03922548 0.4022838
35 0.13173022 0.3966535
39 0.12754208 0.4525918
41 0.21249712 0.4834107
49 0.04747055 0.4845568
53 0.59370349 1.0769781
58 0.20707724 0.3557839
60 0.25196603 0.6378090
65 -0.17490019 0.5684592

Degrees of freedom: 1375 total; 1337 residual
Residual standard error: 0.7329521

and compare the confidence intervals on these coefficients.

> plot(confint(ExamFS, pool = TRUE), order = 1)

> show(ExamMS <- lmList(normexam ~ standLRT | school, Exam, + subset = sex == "M" & type == "Sngl"))

Call: lmList(formula = normexam ~ standLRT | school, data = Exam, subset = sex == "M", + study = exam)

Coefficients:

(Intercept) standLRT
11 0.26596312 0.4586355
24 0.17773174 0.3976156
27 0.03518861 0.5728684
36 -0.20691842 0.4383453
37 -0.48522245 0.2382739

14
Figure 6: Normalized exam scores versus pretest (Standardized London Reading Test) score by school for female students in single-sex schools. School 48 has been eliminated and the panels have been ordered by increasing slope of the within-school regression lines.
Figure 7: Confidence intervals on the coefficients of the within-school regression lines for female students in single-sex schools. School 48 has been eliminated and the schools have been ordered by increasing estimated intercept.

Figure 8: Normalized exam scores versus pretest (Standardized London Reading Test) score by school for male students in single-sex schools.
Figure 9: Confidence intervals on the coefficients of the within-school regression lines for female students in single-sex schools. School 48 has been eliminated and the schools have been ordered by increasing estimated intercept.

The corresponding plot of the confidence intervals is shown in Figure 9.

For the mixed-sex schools we can consider the effect of the pretest score and sex in the plot (Figure 10) and in the separate model fits for each school.
Figure 10: Normalized exam scores versus pretest score by school and sex for students in mixed-sex schools.
Figure 11: Confidence intervals on the coefficients of the within-school regression lines for female students in single-sex schools. School 48 has been eliminated and the schools have been ordered by increasing estimated intercept.

The confidence intervals for these separately fitted models, shown in Figure 11, indicate differences in the intercepts and possibly differences in the slopes with respect to the pretest scores. However, there is not a strong indication of variation by school in the effect of sex.
2.5 Multilevel models for the exam data

We begin with a model that has a random effects for the intercept by school plus additive fixed effects for the pretest score, the student’s sex and the school type.

```r
> (Em3 <- lmer(normexam ~ standLRT + sex + type + (1|school), Exam))
Linear mixed model fit by REML ['lmerMod']
Formula: normexam ~ standLRT + sex + type + (1 | school)
   Data: Exam
REML criterion at convergence: 9345.384
Random effects:
  Groups   Name       Std.Dev.
   school (Intercept) 0.2905
  Residual           0.7500
Number of obs: 4059, groups: school, 65
Fixed Effects:
             (Intercept)    standLRT      sexF typeSngl
           -0.1677         0.5598      0.1660   0.1655
```

Our data exploration indicated that the slope with respect to the pretest score may vary by school. We can fit a model with random effects by school for both the slope and the intercept as

```r
> (Em4 <- lmer(normexam ~ standLRT + sex + type + (standLRT|school), Exam))
Linear mixed model fit by REML ['lmerMod']
Formula: normexam ~ standLRT + sex + type + (standLRT | school)
   Data: Exam
REML criterion at convergence: 9300.573
Random effects:
  Groups   Name       Std.Dev. Corr
   school (Intercept) 0.2872
                     standLRT 0.1228 0.58
  Residual           0.7418
Number of obs: 4059, groups: school, 65
Fixed Effects:
             (Intercept)    standLRT      sexF typeSngl
           -0.1887         0.5541      0.1680   0.1764
```

and compare this fit to the previous fit with

```r
> anova(Em3, Em4)
Data: Exam
Models: Em3: normexam ~ standLRT + sex + type + (1 | school) Em4: normexam ~ standLRT + sex + type + (standLRT | school)
   Df      AIC      BIC    logLik deviance  Chisq Chi Df Pr(>Chisq)
Em3     6 9337.5 9375.3 -4662.7 9325.5
Em4     8 9297.1 9347.6 -4640.6 9281.1  44.327  2 2.368e-10
```

There is a strong evidence of a significant random effect for the slope by school, whether judged by AIC, BIC or the p-value for the likelihood ratio test.
The p-value for the likelihood ratio test is based on a $\chi^2$ distribution with degrees of freedom calculated as the difference in the number of parameters in the two models. Because one of the parameters eliminated from the full model in the submodel is at its boundary the usual asymptotics for the likelihood ratio test do not apply. However, it can be shown that the p-value quoted for the test is conservative in the sense that it is an upper bound on the p-value that would be calculated say from a parametric bootstrap.

Having an upper bound of $1.9 \times 10^{-10}$ on the p-value can be regarded as “highly significant” evidence of the utility of the random effect for the slope by school.

We could also add a random effect for the student’s sex by school

```r
> (Em5 <- lmer(normexam ~ standLRT + sex + type + (standLRT + sex|school), Exam))
```

Linear mixed model fit by REML ['lmerMod']
Formula: normexam ~ standLRT + sex + type + (standLRT + sex | school)
Data: Exam
REML criterion at convergence: 9299.956

Random effects:
Groups  Name         Std.Dev. Corr
school  (Intercept) 0.27553
        standLRT    0.12281 0.62
        sexF        0.02914 0.70 -0.14
Residual         0.74177
Number of obs: 4059, groups: school, 65

Fixed Effects:
(Intercept)  standLRT  sexF  typeSngl
  -0.1895     0.5541  0.1698  0.1762

convergence code 0; 1 optimizer warnings; 0 lme4 warnings

Notice that the estimate of the variance of the sexM term is essentially zero so there is no need to test the significance of this variance component. We proceed with the analysis of Em4.

3 Growth curve model for repeated measures data

```r
> str(Oxboys)
```

data.frame': 234 obs. of 4 variables:
$ Subject : Factor w/ 26 levels "1","10","11",...: 1 1 1 1 1 1 1 1 1 1
$ age     : num -1 -0.7479 -0.463 -0.1643 -0.0027 ...
$ height  : num 140 143 145 147 148 ...
$ Occasion: Factor w/ 9 levels "1","2","3","4",...: 1 2 3 4 5 6 7 8 9 1 ...
- attr(*, "ginfo")=List of 7
  ..$ formula :Class 'formula' language height ~ age | Subject
  ..$ order.groups: logi TRUE
  ..$ FUN        :function (x)

21
> system.time(mX1 <- lmer(height ~ age + I(age^2) + I(age^3) + I(age^4) + (age + I(age^2)|Subject),
+ Oxboys))

user  system elapsed
0.05   0.00   0.05

> summary(mX1)
Linear mixed model fit by REML ['lmerMod']
Formula: height ~ age + I(age^2) + I(age^3) + I(age^4) + (age + I(age^2)|Subject),
    Data: Oxboys
REML criterion at convergence: 627.9
Scaled residuals:
    Min     1Q Median     3Q    Max
-2.71347 -0.54621  0.00289  0.54294  2.65893
Random effects:
 Groups     Name    Variance Std.Dev.   Corr
Subject (Intercept) 64.0943 8.0059
     age  2.8661 1.6930 0.61
     I(age^2) 0.6744 0.8212 0.22 0.66
     Residual 0.2174 0.4662
Number of obs: 234, groups: Subject, 26
Fixed effects:
            Estimate Std. Error   t value
(Intercept) 149.0189    1.5711   94.850
age          6.1742     0.3566   17.314
I(age^2) 1.1282     0.3514    3.210
I(age^3) 0.4539     0.1624    2.794
I(age^4) -0.3769     0.3002   -1.256
Correlation of Fixed Effects:
     (Intr) age I(g^2) I(g^3)
age    0.572
I(age^2) 0.076 0.264
I(age^3) -0.001 -0.340 0.025
I(age^4) 0.021 0.016 -0.857 -0.021

Model failed to converge with max|grad| = 0.00267836 (tol = 0.002, component 1)

> system.time(mX2 <- lmer(height ~ poly(age,4) + (age + I(age^2)|Subject), Oxboys))

user  system elapsed
0.06   0.00   0.06

> summary(mX2)
Linear mixed model fit by REML ['lmerMod']
Formula: height ~ poly(age, 4) + (age + I(age^2) | Subject)
Data: Oxboys

REML criterion at convergence: 616.9

Scaled residuals:
    Min 1Q Median 3Q Max
-2.71340 -0.54607 0.00301 0.54293 2.65879

Random effects:
  Groups   Name       Variance  Std.Dev.  Corr
  Subject (Intercept) 64.0634 8.0040
       age            2.8643 1.6924 0.61
       I(age^2)       0.6742 0.8211 0.22 0.66
  Residual           0.2174 0.4662

Number of obs: 234, groups: Subject, 26

Fixed effects:
            Estimate Std. Error t value
(Intercept) 149.5198   1.5907  93.999
poly(age, 4)1  64.5409   3.3279  19.394
poly(age, 4)2   4.2032   1.0235   4.107
poly(age, 4)3   1.2908   0.4663   2.768
poly(age, 4)4  -0.5855   0.4663  -1.256

Correlation of Fixed Effects:
    (Intr) p(,4)1 p(,4)2 p(,4)3
poly(age,4)1  0.631
poly(age,4)2  0.230  0.583
poly(age,4)3  0.000  0.000  0.000
poly(age,4)4  0.000  0.000  0.000  0.000

4 Cross-classification model

> str(ScotsSec)
'data.frame': 3435 obs. of 6 variables:
$ verbal : num 11 0 -14 -6 -30 -17 -17 -11 -9 -19 ...
$ attain : num 10 3 2 2 2 2 2 2 2 2 ...
$ primary: Factor w/ 148 levels "1","2","3","4"...: 1 1 1 1 1 1 1 1 1 1 ...
$ sex    : Factor w/ 2 levels "M","F": 1 2 1 1 2 2 1 1 1 1 ...
$ social : num 0 0 0 20 0 0 0 0 0 0 ...
$ second : Factor w/ 19 levels "1","2","3","4"...: 9 9 9 9 9 9 1 1 9 9 ...

> system.time(mS1 <- lmer(attain ~ sex + (1|primary) + (1|second), ScotsSec))
user  system elapsed
0.05   0.00   0.04

> summary(mS1)
Linear mixed model fit by REML ['lmerMod']
Formula: attain ~ sex + (1 | primary) + (1 | second)
Data: ScotsSec

REML criterion at convergence: 17127.9
Scaled residuals:

<table>
<thead>
<tr>
<th></th>
<th>Min</th>
<th>1Q</th>
<th>Median</th>
<th>3Q</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>-2.2376</td>
<td>-0.8469</td>
<td>-0.1219</td>
<td>0.9035</td>
<td>2.2001</td>
</tr>
</tbody>
</table>

Random effects:

<table>
<thead>
<tr>
<th>Groups</th>
<th>Name</th>
<th>Variance</th>
<th>Std.Dev.</th>
</tr>
</thead>
<tbody>
<tr>
<td>primary</td>
<td>(Intercept)</td>
<td>1.1096</td>
<td>1.053</td>
</tr>
<tr>
<td>second</td>
<td>(Intercept)</td>
<td>0.3697</td>
<td>0.608</td>
</tr>
<tr>
<td>Residual</td>
<td></td>
<td>8.0551</td>
<td>2.838</td>
</tr>
</tbody>
</table>

Number of obs: 3435, groups: primary, 148; second, 19

Fixed effects:

<table>
<thead>
<tr>
<th></th>
<th>Estimate</th>
<th>Std. Error</th>
<th>t value</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Intercept)</td>
<td>5.25515</td>
<td>0.18432</td>
<td>28.511</td>
</tr>
<tr>
<td>sexF</td>
<td>0.49851</td>
<td>0.09825</td>
<td>5.074</td>
</tr>
</tbody>
</table>

Correlation of Fixed Effects:

<table>
<thead>
<tr>
<th>(Intr)</th>
<th>sexF</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.264</td>
<td></td>
</tr>
</tbody>
</table>

5 Session Info

> toLatex(sessionInfo())

- R version 3.5.2 (2018-12-20), x86_64-w64-mingw32
- Locale: LC_COLLATE=C, LC_CTYPE=English_Canada.1252, LC_MONETARY=English_Canada.1252, LC_NUMERIC=C, LC_TIME=English_Canada.1252
- Running under: Windows 10 x64 (build 17763)
- Matrix products: default
- Base packages: base, datasets, grDevices, graphics, methods, stats, utils
- Other packages: Matrix 1.2-15, lattice 0.20-38, lme4 1.1-20, mlmRev 1.0-8
- Loaded via a namespace (and not attached): MASS 7.3-51.1, Rcpp 1.0.3, compiler 3.5.2, grid 3.5.2, minqa 1.2.4, nlme 3.1-137, nloptr 1.2.1, splines 3.5.2, tools 3.5.2