Deutsch-Sozsa Algorithm

Carsten Urbach

The Deutsch-Sosza Algorithm

This is an example implementation of the Deutsch-Sosza algorithm for the special case of 2 qubits. The algorithm allows to distinguish between a constant or balanced function f with a single application of f, relying on what is called quantum parallelism.

We first prepare a state $|\psi_0\rangle = |x, y\rangle = |0, 1\rangle$ with only 2 qubits as follows

```
x <- X(1) * qstate(nbits=2, basis=genComputationalBasis(2, collapse=","))
x
```

```
##    ( 1 ) * |0,1>
```

Note that we count the qubits from one to number of qubits, and the **least significant bit** (the right most one) is counted **first**.

Using the Hadamard gate on both qubits results in a superposition in both qubits

```
y <- H(2) * (H(1) * x)
y
```

```
##    ( 0.5 )   * |0,0>
##  + ( -0.5 )  * |0,1>
##  + ( 0.5 )   * |1,0>
##  + ( -0.5 )  * |1,1>
```

The next step is to apply the uniform transformation U_f to the state $|x\rangle(|0\rangle - |1\rangle)$. The action of U_f was defined as $|x, y\rangle \to |x, y \oplus f(x)\rangle$, where $\oplus$ is addition modulo 2. The function f is a function $\{0, 1\} \to \{0, 1\}$.

We first consider a so-called balanced function $f(x)$, i.e. it is equal to 1 for exactly half of the possible x. In our case with a single qubit x this could be $f(0) = 0$ and $f(1) = 1$.

U_f is realised in this case by $\mathrm{CNOT}(2, 1)$, where we consider the second qubit as the control qubit. For $|x, y \oplus f(x)\rangle$, there are four different possiblities

- $x = 0, y = 0$, $U_f(|0, 0\rangle) = |0, 0 \oplus f(0)\rangle = |0, 0\rangle$
- $x = 1, y = 0$, $U_f(|1, 0\rangle) = |1, 0 \oplus f(1)\rangle = |1, 1\rangle$
- $x = 0, y = 1$, $U_f(|0, 1\rangle) = |0, 1 \oplus f(0)\rangle = |0, 1\rangle$
- $x = 1, y = 1$, $U_f(|1, 1\rangle) = |1, 1 \oplus f(1)\rangle = |1, 0\rangle$

Now,

- $\mathrm{CNOT}(2, 1)|0, 0\rangle = |0, 0\rangle$
- $\mathrm{CNOT}(2, 1)|1, 0\rangle = |1, 1\rangle$
- $\mathrm{CNOT}(2, 1)|0, 1\rangle = |0, 1\rangle$
- $\mathrm{CNOT}(2, 1)|1, 1\rangle = |1, 0\rangle$

which is what we wanted to archieve. Thus, we apply it:

```
z <- CNOT(c(2, 1)) * y
z
```

```
##    ( 0.5 )   * |0,0>
##  + ( -0.5 )  * |0,1>
##  + ( -0.5 )  * |1,0>
##  + ( 0.5 )   * |1,1>
```

Now apply the Hadamard gate again on x (the query register), i.e. the second qubit

```
u <- H(2) * z
u
```

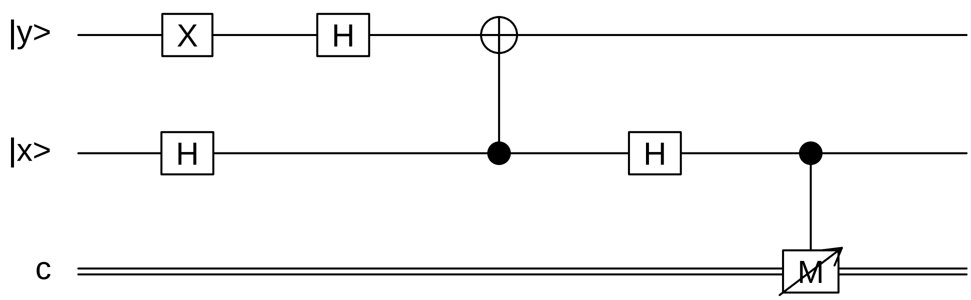
```
##    ( 0.7071068 ) * |1,0>
##  + ( -0.7071068 )    * |1,1>
```

Now qubit 2 equals 1, thus, if we measure,

```
value <- measure(u, 2)$value
```

we obtain 1. We can also plot the corresponding circuit

```
plot(measure(u, 2)$psi, qubitnames=c("|y>", "|x>"), cbitnames="c")
```



On the other hand, a constant function $f(x) = 1$ leads to

- $x = 0, y = 0$, $U_f(|0,0\rangle) = |0, 0 \oplus f(0)\rangle = |0,1\rangle$
- $x = 1, y = 0$, $U_f(|1,0\rangle) = |1, 0 \oplus f(1)\rangle = |1,1\rangle$
- $x = 0, y = 1$, $U_f(|0,1\rangle) = |0, 1 \oplus f(0)\rangle = |0,0\rangle$
- $x = 1, y = 1$, $U_f(|1,1\rangle) = |1, 1 \oplus f(1)\rangle = |1,0\rangle$

which can be realised with a NOT operation on the first qubit

- $\mathrm{X}(1)|0,0\rangle = |0,1\rangle$
- $\mathrm{X}(1)|1,0\rangle = |1,1\rangle$
- $\mathrm{X}(1)|0,1\rangle = |0,0\rangle$
- $\mathrm{X}(1)|1,1\rangle = |1,0\rangle$

So, the same algorithm again, now with the constant f

```
x <- X(1) * qstate(nbits=2, basis=genComputationalBasis(2, collapse=","))
y <- H(2) * (H(1) * x)
z <- X(1) * y
z
```

```
##    ( -0.5 )  * |0,0>
##  + ( 0.5 )   * |0,1>
##  + ( -0.5 )  * |1,0>
##  + ( 0.5 )   * |1,1>
```

```
u <- H(2) * z
u
```

```
##    ( -0.7071068 )    * |0,0>
##  + ( 0.7071068 ) * |0,1>
```

```r
value <- measure(u, 2)$value
```

and we obtain 0 for the second qubit.